

Largest Acylindrical Actions for Artin Groups under a Strict Triple Condition

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Abstract

Let A_S be a finitely generated Artin group such that the sum of the reciprocals of the three labels on every triple of generators is strictly less than one, with $1/\infty = 0$. We record a direct consequence of the Martin–Przytycki coned-off Deligne construction and the domination criterion of Abbott–Balasubramanya–Osin. The coned-off action dominates every acylindrical action on a hyperbolic space precisely when every standard generator has a finite-labelled neighbour. In particular, in that case it is a largest acylindrical hyperbolic structure. Isolated generators obstruct this particular action, but the known free-product combination theorem still gives a largest action for the group. The argument includes mixed label-2 presentations; the large-type overlap is already covered by existing work. This is an attributed corollary note, not a solution of the general AIM question for arbitrary nonspherical irreducible Artin groups.

1 Statement and relation to previous work

Let S be finite, and let A_S have the standard Artin presentation with symmetric labels $m_{st} \in \{2, 3, \dots, \infty\}$. A finite label specifies the equality of the two alternating words of that length; an infinite label specifies no relation. Throughout, assume

$$\frac{1}{m_{st}} + \frac{1}{m_{tr}} + \frac{1}{m_{rs}} < 1 \quad \text{for distinct } s, t, r \in S, \quad \frac{1}{\infty} = 0. \quad (\text{H})$$

Let Γ_f have vertex set S and an edge for each finite label, *including label 2*. Consider the additional condition

$$\text{every vertex of } \Gamma_f \text{ has a neighbour.} \quad (\text{N})$$

We use the explicit hypothesis (H), rather than a broader condition on the associated Coxeter group.

For isometric actions of a group G on X and Y , say that X *dominates* Y if, for base points x, y and some $C \geq 1$,

$$d_Y(y, gy) \leq C d_X(x, gx) + C \quad (g \in G). \quad (3)$$

Write $\mathcal{AH}(G)$ for the equivalence classes of hyperbolic Cayley graphs with acylindrical G -action, ordered as in Abbott–Balasubramanya–Osin [1]. A largest element dominates all such Cayley actions. A structure realizes *strong $\mathcal{AH}$ -accessibility* if it dominates all acylindrical actions on hyperbolic spaces, without requiring the target to be cobounded [1, Definition 7.5]. A *universal* acylindrical action makes every generalized loxodromic element loxodromic; an element is generalized loxodromic

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if it is loxodromic in some acylindrical action on a hyperbolic space. Largest structures are universal [1, 6], but the converse does not hold without further hypotheses.

Under (H), Martin–Przytycki [5] construct a hyperbolic coned-off modified Deligne complex Φ^* with an acylindrical A_S -action. Under (N), their Theorem A proves universality. They also ask whether this is a largest action [5, p. 13102]. The following implication uses their construction and the published domination and combination criteria of [1].

Theorem 1. *Suppose (H) holds.*

1. *If (N) holds, the Martin–Przytycki action on Φ^* realizes strong $\mathcal{AH}$ -accessibility. In particular, it represents the largest element of $\mathcal{AH}(A_S)$.*
2. *If Γ_f has an isolated vertex, this unmodified coned-off action is not universal and is not largest.*
3. *In all cases, A_S is strongly $\mathcal{AH}$ -accessible. A largest Cayley action is obtained from the free-product decomposition over the connected components of Γ_f , using the actions in part 1 for non-singleton components and a Cayley line for each singleton component.*

The statement permits bounded or elementary largest structures. It does not assert that dihedral Artin groups or $\mathbb{Z}$ have non-elementary acylindrical actions. In the irreducible rank-at-least-three case covered by part 1, non-elementarity follows from [5, Theorem A]. The empty presentation is handled by the one-point action of the trivial group.

For large-and-hyperbolic-type Artin groups with connected defining graph, the identification of the natural coned-off Deligne complex as a largest space is already contained in Hagen–Martin–Sisto [4, Corollary B and Lemma 6.18], using Abbott–Behrstock–Durham [2]. Petyt–Spriano–Zalloum explicitly record that identification immediately before their Corollary 7.10 [7]. The present note spells out the direct implication for (H), including mixed label-2 graphs, and the distinction between the unmodified action and the group when isolated generators occur. No new general domination criterion or new Deligne construction is claimed. This restricted statement does not settle AIM Question 4.5 for arbitrary nonspherical irreducible Artin groups [3].

2 A finite quotient and the domination criterion

We recall the published criterion in the form needed here.

Lemma 2 (Abbott–Balasubramanya–Osin, Proposition 4.13 [1]). *Suppose G acts cocompactly on a connected graph Q . If every vertex stabilizer has bounded orbits in every action in a family $\mathcal{A}$, then the action on Q dominates every action in $\mathcal{A}$.*

In our application, it is essential to check finite *edge* orbits after coning off, not just finite vertex orbits. Give the one-skeleton Q of Φ^* unit-length edges. The complex has finitely many shapes of hyperbolic simplices. Its one-skeleton is therefore equivariantly quasi-isometric to the complex: bounded simplex diameters give coarse density and an upper bound on metric length per graph edge; the finite-shape short-gallery bound gives the converse estimate on segments of bounded length, and subdivision of a segment gives the global estimate. This is the same gallery control used in the proof of [5, Lemma 5.1]. Consequently hyperbolicity, acylindricity and loxodromicity transfer between Q and Φ^* .

The original modified Deligne complex has a finite chamber K . Every standard tree is a translate of $T_s = \text{Fix}(s)$ for some $s \in S$. There are thus finitely many orbits of standard trees. By [5, Remark 4.6], $T_s/\text{Stab}(T_s)$ is a component of a finite graph obtained from the graph K_T by

splitting its even-labelled dihedral vertices. In particular, the stabilizer has finitely many orbits of vertices and edges on T_s . Each added cone edge is specified by a standard tree and a vertex of that tree. Its A_S -orbit is determined by one of the finitely many tree orbits and one of the finitely many vertex orbits under the tree stabilizer. Hence Q/A_S has finitely many vertices and edges. Lemma 2 is applicable once ellipticity of its stabilizers has been established.

3 Universal ellipticity of the stabilizers

Assume (N). Let H_v be a vertex stabilizer in Q . Every element of H_v is elliptic on Q . Since the coned-off action is universal by [5, Theorem A], no element of H_v can be loxodromic in any acylindrical action of A_S on a hyperbolic space Y .

Restriction of an acylindrical action to a subgroup remains acylindrical, with the same bounds. Osin's trichotomy [6, Theorem 1.1] implies that every unbounded acylindrical action has a loxodromic element. If H_v had an unbounded orbit in Y , the restricted action would therefore give a loxodromic element in H_v , a contradiction. Thus the *whole subgroup* H_v has bounded orbits. This is stronger than merely asserting that each cyclic orbit is bounded.

Lemma 2, applied to the family of all acylindrical actions on hyperbolic spaces, now proves strong dominance. Since the action on Q is itself cobounded, acylindrical and hyperbolic, it represents an element of $\mathcal{AH}(A_S)$, and part 1 of Theorem 1 follows.

For completeness, the same ellipticity assertion has a direct stabilizer proof that does not use universality. A non-virtually-cyclic group with infinite centre has bounded orbits in every acylindrical action on a hyperbolic space: the virtually cyclic alternative in Osin's trichotomy is excluded, while the non-elementary alternative would contradict [6, Corollary 7.2(a)], which gives finite centre. Up to conjugacy, the vertex stabilizers of Φ^* are

$$1, \quad A_s, \quad A_{st} \ (m_{st} < \infty), \quad \text{Stab}(T_s) = C_{A_S}(s) = \langle s \rangle \times F_s;$$

see [5, Section 3 and Lemma 4.5]. When $m_{st} = 2$, $A_{st} \cong \mathbb{Z}^2$. For $m_{st} \geq 3$, the dihedral calculations in [5, Lemmas 2.1–2.3] give infinite centre and an embedded $\mathbb{Z}^2$. Thus every such A_{st} is universally elliptic, and (N) puts every A_s in one of these groups.

At a cone point, s is an infinite-order central element of $C(s)$. Its order is also detected by the homomorphism taking each standard generator to $1 \in \mathbb{Z}$. Choose a finite-labelled neighbour t . For label 2, $\langle s, t \rangle \cong \mathbb{Z}^2$ lies in $C(s)$; for larger labels, [5, Lemma 2.3] gives $\langle s, z_{st} \rangle \cong \mathbb{Z}^2$ there, where z_{st} is the standard dihedral central element. Hence $C(s)$ is not virtually cyclic and the same criterion applies. In particular, no erroneous inference from infinite centre alone, which would fail for $\mathbb{Z}$, is needed.

One may describe the resulting structure with a relative generating set. Choose finitely many vertex representatives v_i , stabilizers H_i , and a finite symmetric set F of oriented edge-transition elements, enlarged to contain a finite generating set of A_S . Then

$$R = F \cup \bigcup_i H_i \tag{4}$$

has Cayley graph equivariantly quasi-isometric to Q . Indeed, elements of H_i move v_1 a distance at most $2d_Q(v_1, v_i)$, and edge paths give words with boundedly many R -letters per edge, plus an endpoint stabilizer term. Conversely every R -letter has uniformly bounded displacement. For any target acylindrical action, bounded H_i -orbits and finiteness of F give $M = \sup_{r \in R} d_Y(y, ry) < \infty$, so $d_Y(y, gy) \leq M|g|_R$. For a target Cayley generating set U , this is $U \preceq R$ in the order of [1].

4 Isolated generators and free products

If s is isolated in Γ_f , there is no defining relation between s and another generator, and

$$A_S = \langle s \rangle * A_{S \setminus \{s\}}.$$

The discussion following the proof of Theorem A in [5, pp. 13115–13116] provides an acylindrical action in which s is loxodromic: in the tree-of-spaces description, replace the bounded pieces corresponding to conjugates of $\langle s \rangle$ by lines with the translation action. For $S = \{s\}$ use the ordinary Cayley line of $\mathbb{Z}$. In the unmodified Deligne action, s fixes its standard vertex, so that action is not universal. Since a largest acylindrical hyperbolic structure is universal, it is not largest. This proves part 2. In particular, this argument does not use a quotient action with an uncontrolled infinite kernel.

Let $S_1, \dots, S_k$ be the vertex sets of the connected components of Γ_f . All defining relations lie inside components, so

$$A_S = A_{S_1} * \dots * A_{S_k}.$$

Each non-singleton component satisfies (N) and inherits (H); part 1 gives strong $\mathcal{AH}$ -accessibility for its group. A singleton is $\mathbb{Z}$, whose Cayley line is a strongly largest action [1, Example 7.7].

For $k \geq 2$, this free product is hyperbolic relative to its finitely many factors. The strong $\mathcal{AH}$ -accessibility combination theorem [1, Theorem 7.9] applies. Explicitly, let R_i be a generating set realizing the strongly largest action of the i th factor, chosen as in (4) or as a single generator. The induced-action construction [1, Theorem 5.17], used in the proof of their Theorem 7.9, gives an acylindrical hyperbolic Cayley graph for $R = \bigcup_i R_i$. The free factors are strongly hyperbolically embedded relative to the empty relative generating set. The relative Cayley graph using whole factors is the usual hyperbolic tree of cliques for a free product, with its acylindrical action.

For any acylindrical target action of A_S , restriction to a factor remains acylindrical. Strong dominance for that factor bounds displacement of R_i at a fixed target base point. There are only finitely many factors, giving a uniform bound over R , and hence (3). This proves part 3. Using strong, rather than merely cobounded, dominance for the factors is important because their restricted target actions need not be cobounded.

5 Example and scope

The presentation

$$\langle s, t, r \mid stst = tsts, rt = tr \rangle$$

has labels $4, 2, \infty$. Their reciprocal sum is $3/4$, and the finite-edge graph has no isolated vertex. Part 1 applies although the presentation is not of large type. This example is already [5, Example 4.7], not a new construction.

The supplementary standard-library Python checks enumerate 15,756 presentations of ranks one through four over the label alphabet $\{2, 3, 4, 7, \infty\}$, using exact rational arithmetic. They check the hypothesis tests, component splitting and finite quotient-graph bookkeeping, with identical normal and optimized Python results. They do not verify the infinite-group theorems or replace the proof above.

This AI-assisted note is self-audited and unrefereed. It has not received independent review or proof-assistant verification. The result is an explicitly attributed consequence of the cited theorems; the literature search does not certify originality or absolute priority for the mixed-label implication. Groups outside (H), and the full AIM question, remain outside its scope.

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