

Spherical Type Is an Isomorphism Invariant of Artin Groups

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Abstract

We prove that an Artin group abstractly isomorphic to an Artin group of spherical type is itself of spherical type. No condition is imposed on the isomorphism or on the rank of the source presentation. The main observation is that, for an irreducible nonspherical Artin group, the center has zero image in the abelianization. It follows from Digne’s finite-support reflection cocycle and the infinitude of every reflection conjugacy class in an irreducible infinite Coxeter group. For a spherical Artin group, the center injects into the abelianization, and Paris’s commuting-product theorem implies that every nontrivial abstract direct factor has nontrivial center. These properties exclude all nonspherical source components. We also show that the number of spherical irreducible components of a finite-rank Artin presentation is intrinsic to the group: it is the rank of the image of the center in the abelianization. The argument does not assume the general center or $K(\pi, 1)$ conjectures.

1 Introduction

An Artin group is of spherical type when its associated Coxeter group is finite. This definition refers to an Artin presentation, whereas an abstract group isomorphism need not preserve the standard generators, parabolic subgroups, or canonical Coxeter quotient. The following question is Problem 3.1 of the AIM problem list on the geometry and topology of Artin groups [1]: is spherical type invariant under abstract isomorphism? It is also described as open in [2, p. 1508]. We give an affirmative answer.

Theorem 1.1. *Let A_S and B_T be Artin groups, and suppose that the Coxeter group associated to B_T is finite. If $A_S \cong B_T$ as abstract groups, then the Coxeter group associated to A_S is finite.*

Here S is not initially assumed finite. Finite generation of B_T will imply that it is. Paris classified Artin groups already known to be of spherical type up to abstract isomorphism [5]. We use his commuting-product proposition for the spherical target, not his classification theorem to assume sphericity of the source.

The key source-side assertion is weaker than the general center conjecture, which predicts that irreducible nonspherical Artin groups have trivial center. We prove only

$$Z(A_S) \subseteq [A_S, A_S] \quad \text{when } (W_S, S) \text{ is irreducible and infinite.} \quad (1)$$

This is sufficient: spherical centers inject into abelianization, and a nontrivial centerless direct factor cannot occur in a spherical Artin group. The latter fact follows from the central quotients of its irreducible factors, which are centerless, directly indecomposable and have nontrivial torsion.

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The construction underlying (1) is Digne's homomorphism to the semidirect product of the reflection lattice and the Coxeter group [3, Proposition 2.1]. We recall its elementary braid-relation verification. Its first coordinate is a crossed homomorphism and is not assumed injective. The contribution here is the deduction from that construction to the intrinsic center map and then to the unrestricted sphericity statement. No absolute priority claim is made for the intermediate observations or their combination.

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2 Coxeter systems and reflection orbits

Let $(m_{st})_{s,t \in S}$ be a Coxeter matrix: $m_{ss} = 1$, and $m_{st} = m_{ts} \in \{2, 3, \dots\} \cup \{\infty\}$ for $s \neq t$. Write $[x, y]_m$ for the alternating word of length m starting with x . The associated Artin and Coxeter groups are

$$\begin{aligned} A_S &= \langle \sigma_s \ (s \in S) \mid [\sigma_s, \sigma_t]_{m_{st}} = [\sigma_t, \sigma_s]_{m_{st}} \text{ if } m_{st} < \infty \rangle, \\ W_S &= A_S / \langle\langle \sigma_s^2 \ (s \in S) \rangle\rangle. \end{aligned}$$

Let $\pi : A_S \rightarrow W_S$ be the canonical epimorphism, with $\pi(\sigma_s) = s$. The Coxeter graph has an edge between s, t when $m_{st} \neq 2$, including when $m_{st} = \infty$. Its connected components are the irreducible components. For finite S , both groups split as the direct products of their standard component groups.

We use the usual geometric representation of W_S on the real vector space V with basis $(\alpha_s)_{s \in S}$. Its bilinear form satisfies

$$B(\alpha_s, \alpha_s) = 1, \quad B(\alpha_s, \alpha_t) = \begin{cases} -\cos(\pi/m_{st}), & m_{st} < \infty, \\ -1, & m_{st} = \infty. \end{cases}$$

The simple reflection s acts by

$$s(v) = v - 2B(v, \alpha_s)\alpha_s. \tag{2}$$

This is a faithful representation and preserves B ; see [4, pp. 612–613]. Our form is normalized to give each simple root norm one. The form need not be positive definite or nondegenerate. For an infinite index set the vector space is the direct sum, so every vector still has finite support.

Lemma 2.1. *If A_S is finitely generated as an abstract group, then S is finite.*

Proof. Choose finitely many abstract generators and express them as words in the standard letters. These words use a finite subset $U \subseteq S$. Thus the standard letters indexed by U generate A_S , and their images generate W_S . If $s \in S \setminus U$, every $u \in U$ preserves the coefficient of α_s in any vector, by (2). Therefore every word in U does so. But s sends α_s to $-\alpha_s$, a contradiction. This argument does not require injectivity of a standard Artin parabolic subgroup, or faithfulness of the representation in infinite rank. $\square$

Let

$$R_S = \{wsw^{-1} : w \in W_S, s \in S\}$$

be the set of reflections. Conjugation is the action of W_S on R_S .

Lemma 2.2. *If (W_S, S) is irreducible and W_S is infinite, every conjugacy class in R_S is infinite.*

Proof. It suffices to consider a simple reflection s . Suppose its conjugacy class is finite. The negative eigenspace of $ws w^{-1}$ in the geometric representation is the line $\mathbb{R}w\alpha_s$. Every $w\alpha_s$ has B -norm one. A fixed line containing a norm-one vector contains exactly two such vectors, even if B is degenerate. Thus the vector orbit $\mathcal{O} = W_S\alpha_s$ is finite.

Its span $E = \text{span}_{\mathbb{R}} \mathcal{O}$ is W_S -invariant and contains α_s . If $\alpha_t \in E$ and $m_{tu} \neq 2$, then

$$u(\alpha_t) - \alpha_t = -2B(\alpha_t, \alpha_u)\alpha_u \in E,$$

where the coefficient on the right is nonzero. Hence $\alpha_u \in E$. Connectedness of the Coxeter graph gives $E = V$.

An element of W_S fixing every vector in $\mathcal{O}$ fixes its span pointwise, and therefore is the identity by faithfulness. Consequently W_S embeds in the finite permutation group of $\mathcal{O}$, a contradiction. Every reflection is conjugate to a simple reflection, which completes the proof. $\square$

3 The reflection cocycle and the center

Let

$$M_S = \bigoplus_{t \in R_S} \mathbb{Z}e_t$$

be the free abelian group on the reflections. The direct sum is essential: its elements have finite support. Let W_S act by $w \cdot e_t = e_{wtw^{-1}}$, and give $M_S \rtimes W_S$ the multiplication

$$(v, w)(v', w') = (v + w \cdot v', ww').$$

Lemma 3.1 (Digne). *The assignments*

$$F(\sigma_s) = (e_s, s)$$

define a homomorphism $F : A_S \rightarrow M_S \rtimes W_S$ whose second coordinate is π . Writing $F(a) = (N(a), \pi(a))$, one has

$$N(ab) = N(a) + \pi(a) \cdot N(b). \quad (3)$$

Proof. This is [3, Proposition 2.1]. For completeness, consider a finite label $m = m_{st}$. The two alternating products of length m have the same W_S -coordinate, the longest element of the finite dihedral standard parabolic $\langle s, t \rangle$. The first coordinate of a product with letters $s_1, \dots, s_m$ is

$$\sum_{i=1}^m e_{(s_1 \cdots s_{i-1})s_i(s_1 \cdots s_{i-1})^{-1}}.$$

For either alternating expression these are precisely the m distinct dihedral reflections. Thus both first coordinates are their sum, and the braid relation holds. Infinite labels impose no relation. The multiplication law gives (3); it also gives $F(\sigma_s^{-1}) = (-e_s, s)$. $\square$

Let $\mathcal{C}$ be the set of connected components of the graph on S having only the finite odd-labelled edges. The abelianized Artin presentation gives

$$(A_S)_{\text{ab}} \cong \mathbb{Z}^{(\mathcal{C})}, \quad \text{ab}(\sigma_s) = f_{[s]}. \quad (4)$$

Indeed, an odd braid relation identifies its two generators in abelianization, whereas an even relation imposes no condition.

Two simple reflections of W_S are conjugate if and only if they belong to the same member of $\mathcal{C}$. An odd dihedral relation gives conjugacy, and the assertion extends along paths. Conversely, the assignment of each simple reflection to its component vector in $(\mathbb{Z}/2)^{(\mathcal{C})}$ respects every Coxeter relation and separates different components. Since the target is abelian, simple reflections in distinct components cannot be conjugate.

Consequently the assignment

$$\rho : M_S \longrightarrow \mathbb{Z}^{(\mathcal{C})}, \quad \rho(e_{ws^{-1}}) = f_{[s]}, \quad (5)$$

is well defined and W_S -invariant. It follows from (3) that ρN is a homomorphism. It agrees with ab on the standard generators, hence

$$\rho \circ N = \text{ab}. \quad (6)$$

In particular, this assertion does not say that N is a homomorphism or that it factors through abelianization.

Proposition 3.2. *If (W_S, S) is irreducible and infinite, then $Z(A_S) \subseteq [A_S, A_S]$.*

Proof. Take $z \in Z(A_S)$ and write $F(z) = (v, w)$. Surjectivity of π implies $w \in Z(W_S)$, so w acts trivially on every reflection basis vector. Comparing $F(z\sigma_s)$ with $F(\sigma_s z)$ gives

$$(v + e_s, ws) = (e_s + s \cdot v, sw).$$

Thus $s \cdot v = v$ for every $s \in S$, so v is W_S -invariant. By Lemma 2.2 all the orbits in R_S are infinite. An invariant vector with finite support therefore vanishes. Hence $N(z) = 0$, and (6) gives $\text{ab}(z) = 0$. $\square$

Remark 3.3. The proposition does not imply $z = 1$ and does not establish the general center conjecture. Injectivity of F or N is neither true in general nor needed. For example, when A_S is the free group on σ_s, σ_t , the elements $F(\sigma_s^2) = (2e_s, 1)$ and $F(\sigma_t^2) = (2e_t, 1)$ commute. The nontrivial reduced word $[\sigma_s^2, \sigma_t^2]$ therefore lies in $\ker F$. Irreducibility also matters: the central rank-one factor of $\mathbb{Z} \times F_2$ has nonzero image in abelianization.

4 Direct factors of spherical Artin groups

We recall the precise spherical inputs. If G is an irreducible spherical Artin group, it is torsion-free, its center is infinite cyclic with a positive generator δ , and δ is either the Garside element Δ or Δ^2 . A positive power of a Coxeter element $c = \sigma_{s_1} \cdots \sigma_{s_n}$ equals δ . These classical facts are recalled in [5, Sections 2–3 and Theorem 3.3]. We also use the following theorem there:

Theorem 4.1 (Paris, Proposition 4.2). *Let G be an irreducible spherical Artin group. If subgroups H_1, H_2 satisfy $G = H_1 H_2$ and $[H_1, H_2] = 1$, then one of them is contained in $Z(G)$.*

The subgroups in this theorem are arbitrary; they need not themselves be Artin groups. Only the ambient group is assumed spherical.

Lemma 4.2. *For every spherical Artin group B , the map $Z(B) \rightarrow B_{\text{ab}}$ is injective. Equivalently, $Z(B) \cap [B, B] = 1$.*

Proof. For an irreducible spherical factor G , the homogeneous length map $\ell : G \rightarrow \mathbb{Z}$, defined by $\ell(\sigma_s) = 1$, is nonzero on the positive generator δ of $Z(G)$. It is therefore injective on that cyclic center and factors through G_{ab} . For a general B , both center and abelianization split over its finitely many standard irreducible factors. Apply the preceding argument in each coordinate. In particular, it is the family of factorwise length maps, not a single total length map, that rules out cancellation between distinct central factors. $\square$

Lemma 4.3. *Let G be a nonabelian irreducible spherical Artin group. Then $Q = G/Z(G)$ is nontrivial, centerless, directly indecomposable, and contains a nonidentity element of finite order.*

Proof. If a coset $gZ(G)$ is central in Q , then $[g, h]$ lies in $Z(G) \cap [G, G]$ for every $h \in G$. Lemma 4.2 forces $[g, h] = 1$, so $g \in Z(G)$. Thus Q is centerless.

Suppose $Q = U \times V$, and take the full preimages $\tilde{U}, \tilde{V}$ in G . They generate G and their mutual commutators belong to $Z(G) \cap [G, G] = 1$. By Theorem 4.1, one preimage is central, so the corresponding quotient factor is trivial. Hence Q is directly indecomposable.

For the remaining assertion take a Coxeter element $c = \sigma_{s_1} \cdots \sigma_{s_n}$. Here $n \geq 2$ and $\ell(c) = n$. The length of Δ is the number of positive roots. This number is greater than n : for an adjacent pair of simple roots the root $s(\alpha_t) = \alpha_t + 2\cos(\pi/m_{st})\alpha_s$ is positive and nonsimple. Such a pair exists by irreducibility. Hence $\ell(\delta) > n$, so c is not an integral power of δ and is not central. A positive power of c equals δ , so its image in Q is a nonidentity element of finite order. $\square$

Lemma 4.4. *Let $Q = \prod_{j=1}^k Q_j$, where every Q_j is nontrivial, centerless and directly indecomposable. If $Q = X \times Y$ is an internal direct product, then X and Y are complementary full coordinate subproducts of the displayed product.*

Proof. Let X_j, Y_j be the projected images in Q_j . They commute and generate Q_j . Their intersection is central in Q_j and therefore trivial. Thus $Q_j = X_j \times Y_j$. Direct indecomposability implies that one image is Q_j and the other is trivial.

Let I be the set of coordinates with $X_j = Q_j$. Then X is supported on I , and Y on its complement. For any tuple q supported on I , write $q = xy$ using $Q = XY$. In each coordinate outside I , both q and x are trivial, so y is trivial there; it is also trivial on I . Hence $y = 1$ and $q = x \in X$. This proves that X is the full subproduct on I , not merely a subdirect subgroup. The same argument applies to Y . $\square$

Proposition 4.5. *Every nontrivial abstract direct factor of a spherical Artin group has nontrivial center.*

Proof. Write B as the product of its standard irreducible spherical factors. The rank-one factors have trivial central quotient; omit them when writing

$$B/Z(B) = \prod_{j=1}^k Q_j.$$

By Lemma 4.3, the remaining factors meet the hypotheses of Lemma 4.4, and each has nontrivial torsion.

Suppose $B = X \times Y$ and $Z(X) = 1$. Then

$$Z(B) = 1 \times Z(Y), \quad B/Z(B) = X \times (Y/Z(Y)).$$

Lemma 4.4 identifies the image of X with a full coordinate subproduct of the Q_j . If this subproduct were nonempty, it would contain a nonidentity torsion element. But $X \leq B$ is torsion-free. Thus the subproduct is empty and $X = 1$. If $k = 0$, the quotient is trivial and the same conclusion follows directly. $\square$

5 Sphericity and an intrinsic component count

Proof of Theorem 1.1. Let $\varphi : A_S \rightarrow B_T$ be an abstract isomorphism, with B_T spherical. Then B_T is finitely generated, so S is finite by Lemma 2.1. If B_T is trivial, S must be empty: any nonempty Artin presentation admits a surjective homogeneous length map to $\mathbb{Z}$.

Suppose that the Coxeter graph of A_S has a nonspherical irreducible component, and let C be its standard Artin factor. The factor C is nontrivial, again by its homogeneous length map. For $z \in Z(C)$, Proposition 3.2 gives $z \in [C, C]$. Since C is a direct factor of A_S , it follows that

$$z \in Z(A_S) \cap [A_S, A_S].$$

The isomorphism φ preserves the center and the derived subgroup, and Lemma 4.2 makes their intersection trivial in B_T . Thus $z = 1$, proving $Z(C) = 1$.

On the other hand, $\varphi(C)$ is a nontrivial abstract direct factor of the spherical Artin group B_T . This contradicts Proposition 4.5. Every irreducible component of (W_S, S) is therefore spherical. There are finitely many components, so W_S is finite. $\square$

Corollary 5.1. *For any finite-rank Artin group A_S , the number of spherical irreducible components of its Coxeter graph equals*

$$\text{rank im}(Z(A_S) \longrightarrow (A_S)_{\text{ab}}).$$

In particular, this number is invariant under abstract isomorphism.

Proof. Decompose A_S into its standard irreducible factors. Center and abelianization both split over this product. Each nonspherical factor contributes zero to the image by Proposition 3.2. Each spherical factor has infinite cyclic center with nonzero image in its own abelian direct summand, by Lemma 4.2. These images are independent, which proves the formula. $\square$

The corollary counts only spherical components, not all components. Neither the theorem nor its proof identifies the two canonical Coxeter quotient maps through the given isomorphism. No source rank bound, pure-subgroup preservation, cohomological-dimension formula, or $K(\pi, 1)$ conjecture is used.

6 Supplementary exact checks

The accompanying source package contains a portable Python checker for the reflection-cocycle conventions and elementary boundary cases. It verifies the dihedral braid relations for every $2 \leq m \leq 40$, and includes 3,120 signed contextual relation tests, 3,120 crossed-law tests and 3,120 abelianization tests. It also checks central elements and central-quotient orders in these finite dihedral cases. In the infinite-dihedral Coxeter group it tests all 5,461 signed words of length at most six and retains the explicit noninjectivity witness from Remark 3.3.

Root-orbit calculations cover a finite A_3 example and a degenerate affine A_2 form. A separate checker from the earlier dimension-based approach enumerates all 60 subgroups of $S_3 \times S_3$ and its four ordered direct decompositions; it retains the diagonal direct factor of $C_2 \times C_2$ as a warning against omitting centerlessness in Lemma 4.4.

The programs use exact arithmetic and explicit checks that remain active under optimized Python. Their normal and optimized runs have identical mathematical outputs and no failures. The contextual tests are deterministic samples, and the orbit calculations explore finite domains. They do not prove an infinite-orbit statement or replace any part of the general arguments above.

References

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