

Sharp Degree Bounds for Multipartite Lorentzian Coxeter Forms

Alper Ferudun*

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Abstract

Partition at least four vertices into at least two nonempty parts. Assign Coxeter exponent 2 within each part, a fixed exponent $m \geq 5$ between parts, and replace selected cross-part exponents by 3. Suppose each vertex has at most one such replacement neighbor in any other part. If the replacement graph has maximum degree Δ and $(\cos(\pi/m) - 1/2)\Delta < 1$, we prove that the normalized Coxeter form has exactly one negative eigenvalue and no zero eigenvalue, with an explicit lower bound for its positive eigenvalues. The uniform degree caps are 3 for $m = 5$ and 2 for every finite $m \geq 6$; both are sharp. A paired-part construction gives the entire obstruction spectrum. The diagrams avoid A_3 and B_3 , so the established rewriting theorem of Blasco-García, Cumplido, Holt, Morris-Wright and Rees supplies a quadratic word algorithm for each fixed Artin presentation. The examples include nonstar diagrams with unboundedly many exponent-3 pairs. This is a spectral construction and an application of that algorithm, not a new algorithm for general Artin groups.

1 The construction and degree criterion

For a finite set S , a Coxeter matrix is a symmetric matrix $M = (m_{uv})_{u,v \in S}$ with $m_{uu} = 1$ and $m_{uv} \in \{2, 3, \dots\} \cup \{\infty\}$ for $u \neq v$. The Artin group $\text{Art}(M)$ has generating set S and, for each finite m_{uv} , the relation equating the two alternating words of length m_{uv} in u and v . In particular, exponent 2 means commutation. The normalized Coxeter form has matrix

$$G_{uu} = 1, \quad G_{uv} = -\cos(\pi/m_{uv}) \quad (u \neq v).$$

Only finite exponents occur below. We call the form *Lorentzian* when its inertia, in the order positive, negative, zero, is $(|S| - 1, 1, 0)$.

The AIM word-problem list asks for further finite-label Artin examples with Lorentzian standard form in dimension at least three [1]. Its signature convention does not require a compact or finite-volume hyperbolic reflection chamber. The distinction between these conventions is discussed in [4, Section 11]. Our contribution is a degree criterion for a particular multipartite construction, together with sharp examples. The word algorithm used for these groups is already known [2].

Let

$$S = P_1 \sqcup \dots \sqcup P_k, \quad k \geq 2, \quad N = |S| \geq 4,$$

where every part is nonempty. Let H be a simple graph on S whose edges join different parts. Impose the local condition

$$|\{v \in P_j : \{u, v\} \in E(H)\}| \leq 1 \quad (u \in S, 1 \leq j \leq k). \quad (1)$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

For an integer $m \geq 5$, define

$$m_{uv} = \begin{cases} 2, & u, v \text{ in the same part,} \\ 3, & \{u, v\} \in E(H), \\ m, & \text{otherwise,} \end{cases} \quad u \neq v. \quad (2)$$

Write K for the adjacency matrix of the complete multipartite graph on the parts, and also write H for the adjacency matrix of the replacement graph. Put

$$c = \cos(\pi/m), \quad \delta = c - \frac{1}{2} > 0, \quad A = cK - \delta H, \quad G = I - A. \quad (3)$$

The minus sign in A reflects the decrease of an adjacency weight from c to $1/2$ when exponent m is replaced by exponent 3.

Theorem 1.1. *Let Δ be the maximum degree of H . Under (1)–(3), if*

$$\delta\Delta < 1, \quad (4)$$

then $\text{inertia}(G) = (N - 1, 1, 0)$. Every positive eigenvalue of G is at least $1 - \delta\Delta$.

In particular, $\Delta \leq 3$ suffices for $m = 5$, and $\Delta \leq 2$ suffices for every finite $m \geq 6$. Neither integer cap can be increased uniformly within this construction, even with (1) retained.

Corollary 1.2. *Every Artin presentation defined by (2) and (1) admits quadratic-time geodesic reduction and a quadratic-time word algorithm. Thus the presentations satisfying (4) have Lorentzian form and this algorithmic property. The complexity is measured in word length for each fixed presentation.*

The corollary is an application of [2, Theorem 1.1], not a new rewriting theorem. Its algorithmic assertion does not require the degree bound: some diagrams outside that bound have two negative directions, as Section 3 shows.

2 Proof of the signature and algorithm statements

Order the eigenvalues of A decreasingly. We prove $\lambda_2(A) < 1 < \lambda_1(A)$.

2.1 The upper bound on the second eigenvalue

For $x \in \mathbb{R}^S$, write $s_i = \sum_{u \in P_i} x_u$. Then

$$x^T K x = \left(\sum_{i=1}^k s_i \right)^2 - \sum_{i=1}^k s_i^2. \quad (5)$$

In particular, this form is nonpositive on the codimension-one space $U = \{x : \sum_{u \in S} x_u = 0\}$. This is the elementary one-positive-eigenvalue feature of complete multipartite graphs; see also [3] for their spectra. Further,

$$\begin{aligned} |x^T H x| &\leq \sum_{\{u,v\} \in E(H)} (x_u^2 + x_v^2) \\ &\leq \Delta \|x\|^2. \end{aligned}$$

It follows that

$$x^T A x \leq \delta \Delta \|x\|^2 \quad (x \in U).$$

By the variational principle,

$$\lambda_2(A) \leq \delta \Delta < 1. \quad (6)$$

Consequently all but the smallest eigenvalue of G are at least $1 - \delta \Delta > 0$. This argument works for unequal part sizes and uses no definiteness assumption on the restriction of K to U .

2.2 A negative direction for the Coxeter form

Every cross-part entry of A is at least $1/2$. We exhibit a principal submatrix with an eigenvalue greater than 1 in all possible cases.

If $k \geq 4$, choose one vertex from each of four parts. The Rayleigh quotient of the all-ones vector on those four vertices is at least $3/2$. If $k = 3$, one part has at least two vertices because $N \geq 4$. Choose two in that part and one in each of the others. There are five cross-part pairs, and the same quotient is at least $5/4$.

Suppose $k = 2$ and both parts have at least two vertices. Choose two vertices from each part. Condition (1) makes the replacement edges a matching, of size at most two. The sum of the four edge weights is at least $1 + 2c$, so the all-ones quotient is at least $1/2 + c > 1$.

Finally, suppose $k = 2$ and one part is a singleton. Choose that vertex and three vertices of the other part. At most one of the three weights is $1/2$. A weighted three-arm star with weights a_1, a_2, a_3 has largest eigenvalue $\sqrt{a_1^2 + a_2^2 + a_3^2}$, as follows directly by solving its center and leaf equations. Therefore the chosen principal submatrix has largest eigenvalue at least

$$\sqrt{\frac{1}{4} + 2c^2} \geq \sqrt{1 + \frac{\sqrt{5}}{4}} > 1,$$

where $c \geq \cos(\pi/5) = (1 + \sqrt{5})/4$.

Extending the relevant test vector by zero gives $\lambda_1(A) > 1$. Together with (6), this proves the signature and spectral bound in Theorem 1.1. The degree caps and their sharpness are established in Section 3.

2.3 Avoidance of the forbidden triples

Theorem 1.1 of [2] applies when no three standard generators have unordered pair exponents $(2, 3, 3)$ or $(2, 3, 4)$, the types A_3 and B_3 . There is no exponent 4 in (2). In a putative $(2, 3, 3)$ triple, the endpoints of the two exponent-3 pairs would commute and hence belong to the same part. Their common neighbor would violate (1). Thus neither forbidden triple exists, proving Corollary 1.2.

The cited theorem was posted in December 2024, before its 2026 journal publication. We make no uniform bit-complexity assertion when a labeled diagram, with variable binary-encoded exponents, is part of the input.

3 The exact obstruction spectrum

For $r \geq 2$, take r parts $P_i = \{i^+, i^-\}$. Join i^+ to j^- in H exactly when $i \neq j$. Thus H is the bipartite double cover of K_r , regular of degree $r - 1$, and satisfies (1).

Proposition 3.1. *For this paired-part family, the Gram eigenvalues and multiplicities are*

<i>eigenvalue</i>	<i>multiplicity</i>
$1 - (r - 1)(c + 1/2)$	1
$1 - (r - 1)\delta$	1
$1 + c + 1/2$	$r - 1$
$1 + \delta$	$r - 1$.

In particular the family is Lorentzian precisely when $(r - 1)\delta < 1$; equality gives one negative and one zero eigenvalue, and $(r - 1)\delta > 1$ gives exactly two negative eigenvalues.

Proof. Split $\mathbb{R}^{2r}$ into the r -dimensional even and odd subspaces given by $x_{i+} = x_{i-}$ and $x_{i+} = -x_{i-}$, respectively. For each distinct pair of parts the adjacency block is

$$\begin{pmatrix} c & 1/2 \\ 1/2 & c \end{pmatrix}.$$

Hence A acts on the two subspaces as $(c + 1/2)(J_r - I_r)$ and $\delta(J_r - I_r)$, respectively. The constant vector and the zero-sum subspace give eigenvalues $r - 1$ and -1 of $J_r - I_r$, with multiplicities 1 and $r - 1$. Subtraction from I gives the table. Since $r \geq 2$ and $c > 1/2$, the first entry is negative and the final two are positive. The remaining entry proves the inertia assertions. $\square$

For $m = 5$, we have $\delta = (\sqrt{5} - 1)/4$, and

$$3\delta < 1 < 4\delta.$$

Thus the sufficient uniform degree cap is 3. At $r = 4$, the paired-part diagram has eight generators and inertia $(7, 1, 0)$. Its smallest positive eigenvalue is exactly $1 - 3\delta$, attaining the bound in Theorem 1.1. At $r = 5$, the ten-generator diagram has degree 4 and inertia $(8, 2, 0)$. This disproves the uniform extension to degree 4.

For every finite $m \geq 6$, we have $c < 1$, so $2\delta < 1$. On the other hand,

$$3\delta \geq \frac{3(\sqrt{3} - 1)}{2} > 1,$$

where $\sqrt{3} > 5/3$ follows by squaring. The uniform cap is therefore 2: $r = 3$ is Lorentzian and attains the positive-eigenvalue bound, whereas $r = 4$ has exactly two negative directions. This completes the sharpness assertion of Theorem 1.1.

These two-negative examples still avoid A_3 and B_3 and still admit the established quadratic word algorithm. They are obstructions to enlarging our signature criterion, not counterexamples to that algorithm or to an AIM conjecture.

4 Explicit families and scope

Taking parts of sizes 1 and $N - 1$, with exactly one replacement edge, gives a star example of inertia $(N - 1, 1, 0)$ for every $N \geq 4$. The general criterion also permits nonstar diagrams and arbitrarily many exponent-3 pairs.

For example, fix $m = 5$ and take four parts of equal size $t \geq 2$, writing their vertices as (i, a) with $1 \leq i \leq 4$ and $1 \leq a \leq t$. Put an edge in H between (i, a) and (j, a) for every $i \neq j$. Then H is the disjoint union of t copies of K_4 , each meeting every part once. Each vertex has degree

3 and one neighbor in each other part. The resulting Coxeter form has inertia $(4t - 1, 1, 0)$, and Corollary 1.2 applies.

The displayed standard diagrams have exponent-3 pairs and commuting pairs. They are therefore neither 3-free nor large type. They are not of FC type: every pair exponent is finite, whereas the full form is indefinite. The triple $(1, 1), (1, 2), (2, 1)$ has pair exponents $(2, 3, 5)$ and is spherical of type H_3 . Thus these diagrams are not locally reducible either. These statements concern the displayed standard presentations; no exclusion of an alternative presentation or pairwise nonisomorphism of the abstract groups is asserted.

The local condition and $N \geq 4$ are substantive. Without (1), the three-arm star with every arm labeled 3 has positive-definite form of type D_4 . With only three vertices, the $(2, 3, 5)$ diagram is itself positive definite. Neither lies within the theorem's hypotheses.

The construction concerns Lorentzian signature, not word-hyperbolicity of an Artin group or compactness of a reflection chamber. In particular, it does not answer the more restrictive historical small-type question in [4, Remark 11.3]: small type uses only pair exponents 2 and 3, whereas our nonstar examples use exponent 5. Nor does this construction settle the word problem for arbitrary Artin groups or exhaust an open-ended request for examples.

5 Exact checks and reproducibility

The accompanying program uses rational pairs for $\mathbb{Q}(\sqrt{5})$, not floating-point eigenvalues. It exhausts the cross-part replacement subsets for each integer partition of four and five vertices. Exactly 1190 diagrams satisfy the local and degree conditions, and their forms have the stated inertia. Across all 2120 subsets, local admissibility agrees with avoidance of $(2, 3, 3)$. Eight paired-part cases, $r = 3, \dots, 10$, pass exact inertia checks and 104 eigenvector identities. The four-part examples with $t = 2, 3, 4$ and the D_4 negative control are also checked. Normal and optimized Python runs give identical reports.

These finite checks supplement the arbitrary-rank proofs; they do not replace them or implement the cited word algorithm. The source archive contains the program, outputs and verification report. The present manuscript is unrefereed and makes no absolute priority claim. The author used AI assistance in developing the construction, checking the arguments, consulting sources and preparing the text. The recorded assessment is a self-audit, not independent peer review or formal proof-assistant verification.

References

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