Free-by-Cyclic Groups with Unboundedly Many BNS Component Orbits

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Abstract

An AIM problem asks for families of free-by-cyclic groups with first Betti number greater than two and many connected components of the Bieri–Neumann–Strebel invariant, even after quotienting by the outer automorphism group. For every $m \geq 2$ we construct a linearly growing UPG automorphism of F_{2m+1} whose mapping torus G_m has $b_1(G_m) = m + 2$ and whose BNS invariant is the complement of the $m + 1$ character hyperplanes $x + iy = 0$, $0 \leq i \leq m$. It consequently has exactly $2m + 2$ connected components. For each component C, we minimize the rank of the kernel of a primitive integral character in C. This minimum is invariant under the full $\mathrm{Out}(G_m)$ action. Its $m + 1$ distinct values prove that the components form at least $m + 1$ outer-automorphism orbits. The construction therefore answers the AIM request in the quantitatively unbounded sense. This note is unrefereed and makes no absolute priority claim.

1 Introduction

For a finitely generated group G, write

$$S(G) = \big(H^1(G; \mathbb{R}) \setminus \{0\}\big)/\mathbb{R}_{>0}$$

for the positive character sphere and $\Sigma^1(G) \subset S(G)$ for the Bieri–Neumann–Strebel invariant [BNS87]. Since $\Sigma^1(G)$ is characteristic, $\mathrm{Out}(G)$ acts on its set of connected components.

Problem 6.2 from the 2023 AIM workshop on rigidity properties of free-by-cyclic groups asks for families of free-by-cyclic groups G such that $b_1(G) > 2$ and $\Sigma^1(G)$ has many components up to the action of $\mathrm{Out}(G)$ [Ame23a, Ame23b]. The phrase "many components" is deliberately qualitative. We prove the strongest natural family statement: both the number of components and the number of their full outer-automorphism orbits tend to infinity.

Our construction is an elementary chorded variation on the graph-of-groups examples in Cashen–Levitt [CL16]. Their BNS formula identifies the relevant hyperplane complement. The additional ingredient here is an intrinsic component statistic: the least rank of a free kernel represented by a primitive integral character in that component. It distinguishes an unbounded number of component orbits without requiring an automorphism to preserve the displayed splitting.

Theorem 1.1. *For every integer $m \geq 2$, there is a linearly growing UPG automorphism $\phi_m \in \mathrm{Aut}(F_{2m+1})$ whose mapping torus $G_m = F_{2m+1} \rtimes_{\phi_m} \mathbb{Z}$ satisfies the following.*

1. $b_1(G_m) = m + 2$.

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2. *In integral character coordinates* $(x, y, z_0, z_2, \ldots, z_m) \in \mathbb{Z}^{m+2}$,

$$\Sigma^1(G_m) = \{[\chi] \in S(G_m) : x + iy \neq 0 \text{ for } 0 \leq i \leq m\}. \tag{1.1}$$

3. *The invariant* $\Sigma^1(G_m)$ *has exactly* $2m+2$ *connected components, or* $m+1$ *antipodal pairs.*

4. *These components form at least* $m+1$ *orbits under the full group* $\mathrm{Out}(G_m)$*.*

The theorem supplies an affirmative and quantitatively unbounded answer to the AIM problem. We do not determine the exact number of orbits, nor whether the two members of every antipodal pair are exchanged.

2 The graph automorphism

Fix $m \geq 2$. Let Γ_m have vertices $v_0, \ldots, v_m$, with a loop a_i at v_i. Add path edges

$$e_i : v_{i-1} \longrightarrow v_i \quad (1 \leq i \leq m),$$

a closing edge $e_0 : v_m \to v_0$, and chords

$$c_i : v_0 \longrightarrow v_i \quad (2 \leq i \leq m).$$

There are $m+1$ vertices and $3m+1$ edges, so

$$\operatorname{rank} \pi_1(\Gamma_m) = (3m+1) - (m+1) + 1 = 2m+1. \tag{2.1}$$

Fix every vertex and every loop and define a graph map f_m by

$$\begin{aligned} f_m(e_i) &= e_i a_i && (1 \leq i \leq m), \\ f_m(e_0) &= e_0 a_0, && \\ f_m(c_i) &= c_i a_i^i && (2 \leq i \leq m). \end{aligned}$$

Lemma 2.1. *The map* f_m *is a homotopy equivalence inducing a genuinely linearly growing UPG outer automorphism of* F_{2m+1}*.*

Proof. For every connecting edge, replacing its fixed-loop suffix by the inverse suffix defines a graph map inverse after tightening. Thus f_m is a homotopy equivalence. Every connecting edge d has $f_m^q(d) = da^{qp}$ for a fixed loop a and an integer $p \geq 1$, whereas each a_i is fixed. The edge filtration with all fixed loops first and each connecting edge in a one-edge stratum is therefore an upper-triangular polynomial representative, so the induced outer automorphism is UPG and has growth at most linear.

It is not boundedly growing. On homology, $N = (f_m)_* - I$ sends connecting cycles into the span of the fixed loop classes and kills that span, hence $N^2 = 0$. The cycle $e_1 \cdots e_m e_0$ acquires the nonzero class $[a_0] + \cdots + [a_m]$. Thus $N \neq 0$, and the growth is genuinely linear. □

Choose a basepoint and let $\phi_m \in \mathrm{Aut}(F_{2m+1})$ represent the induced outer automorphism. Its mapping torus does not depend, up to isomorphism, on this choice.

3 A graph-of-groups presentation

The mapping torus is a graph of groups over the connecting graph. At v_i the vertex group is

$$V_i = \langle a_i, t_i \mid [a_i, t_i] = 1 \rangle \cong \mathbb{Z}^2. \tag{3.1}$$

If $d : u \to v$ satisfies $f_m(d) = da_v^p$, its mapping-torus square gives

$$d^{-1} t_u d = t_v a_v^{-p}. \tag{3.2}$$

Choose $e_1, \ldots, e_m$ as a maximal tree, and denote the stable letters for $e_0, c_2, \ldots, c_m$ by $s_0, s_2, \ldots, s_m$. We obtain

$$\begin{aligned} G_m = \langle a_i, t_i \ (0 \le i \le m),\ s_0, s_2, \ldots, s_m \mid\ & [a_i, t_i] = 1, \\ & t_{i-1} = t_i a_i^{-1} \ (1 \le i \le m), \\ & s_0^{-1} t_m s_0 = t_0 a_0^{-1}, \\ & s_i^{-1} t_0 s_i = t_i a_i^{-i} \ (2 \le i \le m) \rangle. \end{aligned} \tag{3.3}$$

Every cyclic edge group has infinite index in both adjacent $\mathbb{Z}^2$ vertex groups. The finite graph of groups is therefore reduced; since it has multiple vertices, it is not an ascending HNN extension.

4 Characters and the BNS invariant

Let $x_i = \chi(t_i)$ and $y_i = \chi(a_i)$. Abelianizing the path-edge relations in (3.3) gives

$$x_i = x_{i-1} + y_i. \tag{4.1}$$

The ith chord relation gives

$$x_i = x_0 + i y_i \qquad (2 \le i \le m). \tag{4.2}$$

Induction in (4.1)–(4.2) yields

$$y_1 = \cdots = y_m =: y, \qquad x_i = x_0 + iy. \tag{4.3}$$

Finally the closing edge gives

$$y_0 = x_0 - x_m = -my. \tag{4.4}$$

Write $x = x_0$. The values $z_0, z_2, \ldots, z_m$ on the m stable letters are unrestricted. These equations hold integrally, not just over $\mathbb{R}$, and give

$$H^1(G_m; \mathbb{R}) \cong \mathbb{R}^{m+2}, \qquad \chi \longleftrightarrow (x, y, z_0, z_2, \ldots, z_m). \tag{4.5}$$

In particular $b_1(G_m) = m + 2$.

We next compute $\Sigma^1(G_m)$. Virtually polycyclic groups have full BNS invariant. Cashen–Levitt's graph-of-groups criterion [CL16, Corollary 2.10] therefore applies to the reduced, nonascending graph of groups above: a character class lies in $\Sigma^1(G_m)$ if and only if the character is nonzero on every edge group.

The m path edges have source generators $t_0, \ldots, t_{m-1}$ and values $x, x + y, \ldots, x + (m-1)y$. The closing edge has source generator t_m and value $x + my$. The $m - 1$ chords have source generator t_0 and contribute further copies of x. Hence the nonvanishing criterion is precisely (1.1).

The excluded hyperplanes are inverse images, under the projection

$$H^1(G_m; \mathbb{R}) \longrightarrow \mathbb{R}^2, \qquad \chi \longmapsto (x, y), \tag{4.6}$$

of the $m+1$ distinct central lines $x+iy=0$. Their complement in $\mathbb{R}^2$ consists of $2m+2$ open sectors. The inverse image of each sector is its product with the m-dimensional z-space, hence an open convex cone. Positive projectivization gives exactly $2m+2$ connected components of $\Sigma^1(G_m)$.

Let $O^{\pm}$ denote the antipodal pair of outer sectors, where all $x+iy$ have one sign. For $0 \le j \le m-1$, let C_j^+ be the sector

$$y>0, \qquad x+iy<0 \ (0 \le i \le j), \qquad x+iy>0 \ (j+1 \le i \le m), \tag{4.7}$$

and put $C_j^- = -C_j^+$. These are all the components.

5 Free-kernel ranks

The multiplicities suppressed in the hyperplane complement become important for the following exact rank formula.

Proposition 5.1. *Let $\chi : G_m \twoheadrightarrow \mathbb{Z}$ be primitive with $[\chi] \in \Sigma^1(G_m)$, and put $K = \ker \chi$. Then K is free and*

$$\operatorname{rank} K = 1 + m|x| + \sum_{i=1}^{m} |x+iy|. \tag{5.1}$$

Proof. Let T be the Bass–Serre tree of (3.3). Because χ is nonzero on every cyclic edge group, K has trivial edge stabilizers. Its vertex stabilizers are the kernels of nonzero homomorphisms $\mathbb{Z}^2 \to \mathbb{Z}$, and are therefore infinite cyclic.

For an edge group $G_e = \langle g_e \rangle$, normality of K gives

$$|K \backslash G_m / G_e| = [\chi(G_m) : \chi(G_e)] = |\chi(g_e)|. \tag{5.2}$$

Thus $K\backslash T$ is finite. If it has E edges and V vertices, its graph of groups has V infinite-cyclic vertex groups and trivial edge groups. Consequently

$$K \cong (*_V \mathbb{Z}) * F_{E-V+1} \cong F_{E+1}. \tag{5.3}$$

Summing (5.2) over the full edge-orbit table gives m copies of $|x|$ and one copy of $|x+iy|$ for each $1 \le i \le m$, proving (5.1). $\square$

For $(x,y)=(1,0)$, (5.1) gives $2m+1$, as it must for the original free kernel $\pi_1(\Gamma_m)$.

6 Separating full outer-automorphism orbits

For a component C of $\Sigma^1(G_m)$ define

$$\mu(C) = \min\{\operatorname{rank}(\ker \chi) : \chi : G_m \twoheadrightarrow \mathbb{Z} \text{ primitive}, \ [\chi] \in C\}. \tag{6.1}$$

Every chamber is rational and contains primitive integral characters, and the displayed set consists of positive integers, so the minimum exists.

Lemma 6.1. *The function μ is constant on orbits of components under the full group* $\operatorname{Out}(G_m)$.

Proof. If $\alpha \in \operatorname{Aut}(G_m)$, then

$$\ker(\chi \circ \alpha) = \alpha^{-1}(\ker \chi) \cong \ker \chi. \tag{6.2}$$

Pullback by α bijects primitive integral characters, preserves $\Sigma^1(G_m)$, and permutes its components. It therefore preserves the minimum in (6.1). Inner automorphisms act trivially on characters, so the conclusion descends to $\operatorname{Out}(G_m)$. No preservation of the displayed splitting is assumed. $\square$

We compute the invariant. On $O^{\pm}$, all $2m$ integer summands after the initial 1 in (5.1) are nonzero, so each has absolute value at least one. Equality is attained at $(x, y) = (\pm 1, 0)$, whence

$$\mu(O^{\pm}) = 1 + 2m. \tag{6.3}$$

For C_j^+, write $x = -p$ and $y = q$. Its inequalities are

$$q > 0, \qquad jq < p < (j+1)q. \tag{6.4}$$

For integral p, q this forces $q \geq 2$. Put $M = m(m+1)/2$. Removing the initial 1 from (5.1) gives

$$\begin{aligned} N_j(p,q) &= mp + \sum_{i=1}^{j}(p - iq) + \sum_{i=j+1}^{m}(iq - p) \\ &= 2jp + q\big(M - j(j+1)\big). \end{aligned} \tag{6.5}$$

For fixed q, the least permitted p is $jq + 1$; if $j = 0$, the expression does not depend on p. Therefore

$$N_j(p,q) \geq q\big(M + j(j-1)\big) + 2j \geq 2M + 2j^2. \tag{6.6}$$

Equality is attained at $(p, q) = (2j + 1, 2)$. These two integers are coprime, so the corresponding character is primitive. Negating a character preserves its kernel, and hence

$$\mu(C_j^{\pm}) = 1 + m(m+1) + 2j^2 \qquad (0 \leq j \leq m-1). \tag{6.7}$$

The m values in (6.7) are pairwise distinct. For $m \geq 2$ their least value $1 + m(m+1)$ is also strictly greater than $1+2m$. Thus the components carry at least $m+1$ distinct values of the full automorphism invariant μ. They belong to at least $m + 1$ $\mathrm{Out}(G_m)$-orbits, completing the proof of theorem 1.1.

7 Verification, scope, and prior work

The abelian relation matrix and every chamber minimum were independently checked by exact rational row reduction and exhaustive integer enumeration for $2 \leq m \leq 30$. The computation is a regression check, not a substitute for the symbolic proof. The source archive includes the checker and its instructions.

Cashen–Levitt's cycle-of-tori example already shows that polynomially growing free-by-cyclic groups can require arbitrarily many excluded rational subspheres [CL16, Example 5.13]. Polytope descriptions of BNS invariants and fibered norms appear in [FK18, Kie20]; outer automorphism groups of linearly growing mapping tori are studied in [AM22]. We did not locate in these sources the chord family, the minimum-rank values (6.7), or an unbounded lower bound for component orbits. An anonymous public candidate released after the initial preparation of this manuscript records the same weighted-chord family and the same component and orbit bounds [Ano26]. The present manuscript therefore makes no novelty or public-priority claim for this construction.

The groups constructed here have linearly growing monodromy and contain many $\mathbb{Z}^2$ subgroups. No hyperbolic or fully irreducible analogue is claimed. The proof gives a lower bound on the number of component orbits, not their exact number.

AI-assisted tools were used for source discovery, algebraic exploration, exact-check code, adversarial proof review, and preparation of this manuscript. Every mathematical claim used in the theorem is accompanied by an explicit argument or a cited theorem. The named author is responsible for the final manuscript.

Data and code availability. The exact checker and a verification report accompany the source. No external data are required.

References

[AM22] Naomi Andrew and Armando Martino. Free-by-cyclic groups, automorphisms and actions on nearly canonical trees. *Journal of Algebra*, 604:451–495, 2022. `https://arxiv.org/abs/2106.02541`.

[Ame23a] American Institute of Mathematics. Rigidity properties of free-by-cyclic groups. Workshop page, `https://aimath.org/pastworkshops/freebycyclic.html`, 2023. Workshop held October 2–6, 2023; accessed August 29, 2026.

[Ame23b] American Institute of Mathematics. Workshop report: Rigidity properties of free-by-cyclic groups. `https://aimath.org/pastworkshops/freebycyclicrep.pdf`, 2023. Problem 6.2 in the BNS invariants section; accessed August 29, 2026.

[Ano26] Anonymous. A linearly growing free-by-cyclic family with unbounded BNS component-orbit count, August 2026. Version 0.2.0-candidate; unrefereed evidence package; `https://doi.org/10.5281/zenodo.22201487`.

[BNS87] Robert Bieri, Walter D. Neumann, and Ralph Strebel. A geometric invariant of discrete groups. *Inventiones Mathematicae*, 90(3):451–477, 1987. `https://doi.org/10.1007/BF01389175`.

[CL16] Christopher H. Cashen and Gilbert Levitt. Mapping tori of free group automorphisms, and the Bieri–Neumann–Strebel invariant of graphs of groups. *Journal of Group Theory*, 19(2):191–216, 2016. `https://doi.org/10.1515/jgth-2015-0038`.

[FK18] Florian Funke and Dawid Kielak. Alexander and thurston norms, and the Bieri–Neumann–Strebel invariants for free-by-cyclic groups. *Geometry & Topology*, 22(5):2647–2696, 2018. `https://arxiv.org/abs/1605.09067`.

[Kie20] Dawid Kielak. The Bieri–Neumann–Strebel invariants via newton polytopes. *Inventiones Mathematicae*, 219(3):1009–1068, 2020. `https://arxiv.org/abs/1802.07049`.