

A Compact Cubulation of a Quadratic Free-by-Cyclic Group

Alper Ferudun*

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Abstract

Let ϕ be the automorphism of $F(a, b, c)$ given by $\phi(a) = a$, $\phi(b) = ab$, and $\phi(c) = bcb$. We construct a compact nonpositively curved cube complex with fundamental group $F(a, b, c) \rtimes_{\phi} \mathbb{Z}$. It has one vertex, seven edges, eight squares, and two three-dimensional cubes. Its universal cover therefore gives a free, proper, cocompact action on a three-dimensional CAT(0) cube complex. The construction uses a two-square complex, a four-vertex cyclic core that maps locally isometrically into it, and a cubical involution exchanging the attaching data. All link and presentation calculations are explicit. The result concerns the formula $b \mapsto ab$; it does not address the different palindromic formula $b \mapsto aba$ in the associated workshop report.

1 Statement and source scope

We use the mapping-torus convention

$$G = \langle a, b, c, t \mid tat^{-1} = a, tbt^{-1} = ab, tct^{-1} = bcb \rangle. \quad (1)$$

The substitution defining ϕ is invertible: its inverse sends a to a , b to $a^{-1}b$, and c to $b^{-1}acb^{-1}a$.

Theorem 1. *The group G in (1) is the fundamental group of a compact nonpositively curved cube complex Y with cell counts*

$$(f_0, f_1, f_2, f_3) = (1, 7, 8, 2).$$

Consequently G acts freely, properly, and cocompactly on the three-dimensional CAT(0) cube complex $\tilde{Y}$.

The formula in (1) is the literal formulation of record AIM-GEOMETRIC_GROUP_THEORY-0024 in the frozen *UnsolvedMath* version 1.6.0 corpus [6]. That record identifies its source as Problem 5.2 of the AIM list *Rigidity properties of free-by-cyclic groups* [4]. The workshop report instead discusses $a \mapsto a$, $b \mapsto aba$, $c \mapsto bcb$, and describes a three-dimensional candidate cubulation [5]. These formulas must not be conflated. The theorem answers the literal frozen ab formulation and makes no claim about the report's aba example or its intended identification with the list.

For context, Lyman's constructions give geometric actions on CAT(0) spaces for families including the palindromic example, but those spaces need not be cubical [3]. The corrected Hagen–Wise results give free cubical actions under suitable hypotheses and distinguish this from cocompact cubulation [2]. Their Remark 3.7 records the standard compact graph-of-cube-complexes construction when the edge maps are local isometries. We provide the finite attaching complex and the compatible maps for (1), and verify the resulting links directly.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: AlperTheKing.

Throughout, a cube complex uses the usual quotient-cell convention: different faces of a cube may have identified images, as in a cubical torus, while their germs are distinguished in a link. All cubes are unit Euclidean cubes. We use the flag-link criterion and the CAT(0) universal-cover theorem [1, Chapter II.5]. This unrefereed, AI-assisted note presents a self-audited construction; no proof-assistant certification or absolute-priority claim is made.

2 The vertex complex and its involution

Let K have one vertex, three directed edges x, y, b , and two squares with boundary words

$$xyx^{-1}y^{-1}, \quad bxb^{-1}y^{-1}. \quad (2)$$

Thus

$$H = \pi_1(K) = \langle x, y, b \mid [x, y] = 1, bxb^{-1} = y \rangle.$$

Lemma 2. *The complex K is nonpositively curved. The assignments*

$$\psi(x) = y, \quad \psi(y) = x, \quad \psi(b) = b^{-1}$$

define a cubical involution of K .

Proof. The six vertices of Lk_K are the signed edge germs $x^\pm, y^\pm, b^\pm$. The first square gives the four $x^\epsilon y^\delta$ edges, where $\epsilon, \delta \in \{+, -\}$. The second gives

$$b^- - x^+, \quad b^- - x^-, \quad b^+ - y^-, \quad b^+ - y^+.$$

This graph is $K_{3,3}$ with the b^+b^- edge removed, with bipartition $\{x^+, x^-, b^+\}$ and $\{y^+, y^-, b^-\}$. It has no loops, parallel edges, or triangles, and hence is flag.

The substitution ψ sends each word in (2) to a cyclic rotation of the inverse boundary word of the same square. It therefore extends over both squares by cubical isometries. It is an involution on edges, and its square restrictions are involutions as well. $\square$

3 A cyclic core with full links

Define a square complex S with four vertices A, B, C, D and six directed edges as follows. The last column specifies a map to K .

Edge	Initial vertex	Terminal vertex	Label
e_1	A	B	x
e_2	A	C	y
e_3	C	D	x
e_4	B	D	y
e_5	B	A	b
e_6	D	B	b

Attach squares along

$$e_1e_4e_3^{-1}e_2^{-1}, \quad e_5e_1e_6^{-1}e_4^{-1}. \quad (3)$$

Their label words are exactly those in (2), so the label map extends to a cubical map $f : S \rightarrow K$.

Lemma 3. *The map f is a local isometry. The complex S collapses to the two-edge circle e_5e_1 based at B . The images of its positive fundamental-group generator under f and ψf are, respectively,*

$$V = bx, \quad W = b^{-1}y = xb^{-1}.$$

Proof. The images of the four links are listed below. Each signed germ in a row occurs only once.

Vertex	Link vertices	Link edges
A	x^+, y^+, b^-	x^+y^+, b^-x^+
B	x^-, y^+, b^+, b^-	x^-y^+, y^+b^+, x^-b^-
C	y^-, x^+	y^-x^+
D	x^-, y^-, b^+	x^-y^-, y^-b^+

The edge lists have no repetitions and are exactly the edges of Lk_K between the indicated vertices. Thus each link map is injective with full image, proving the local-isometry assertion.

For the collapse, remove the first square across its free edge e_2 , then remove the leaf edge e_3 and vertex C . Remove the second square across its free edge e_6 , then remove the leaf edge e_4 and vertex D . The remaining circle consists of e_5 followed by e_1 . Its labels under the two maps are bx and $b^{-1}y$. The relation $bx b^{-1} = y$ gives $b^{-1}y = xb^{-1}$. $\square$

In particular $\pi_1(S) \cong \mathbb{Z}$, and both maps are π_1 -injective. Besides the local-isometry argument, injectivity follows directly from the homomorphism $H \rightarrow \mathbb{Z}$ taking b to 1 and x, y to 0: it takes V and W to 1 and -1 .

4 Attaching the product and checking curvature

Form

$$Y = K \cup_{f, \psi f} (S \times [0, 1]), \quad (4)$$

where the bottom is attached by f and the top by ψf . Every vertex of Y is the vertex of K . The four vertices of S contribute four interval edges; its six edges contribute six new squares; and its two squares contribute two three-cubes. Together with the cells of K , this gives $(1, 7, 8, 2)$.

Lemma 4. *The vertex link of Y is simplicial and flag. It has 14 vertices, 32 edges, and 16 triangles.*

Proof. Begin with the six-vertex link of K . For each $v \in \{A, B, C, D\}$ there are two additional link vertices, the bottom and top germs of the interval edge at v . At the bottom germ attach the cone on $f(\text{Lk}_S(v))$; at the top germ attach the cone on $\psi f(\text{Lk}_S(v))$. The table in Lemma 3 shows that each base of a cone is an embedded full subcomplex of Lk_K . These are exactly the link simplices furnished by the product cells. Distinct interval germs remain distinct even though their endpoints map to the same vertex of Y .

There is no edge between two interval germs: each product cube has only one interval direction. Consequently a clique in the link contains at most one interval germ. A clique containing none spans a simplex in the flag link of K . If it contains one, all its other vertices lie in the corresponding full sublink, and hence it spans a simplex of the attached cone. This also shows that no additional higher-dimensional simplex is required. Link injectivity in Lemma 3 excludes repeated edges or triangles within a cone; distinct apices exclude repetitions between cones.

There are $6 + 2 \cdot 4 = 14$ link vertices. The total number of vertices of the four links of S is $3 + 4 + 2 + 3 = 12$, and their total number of edges is $2 + 3 + 1 + 2 = 8$. Thus the link of Y has $8 + 2 \cdot 12 = 32$ edges and $2 \cdot 8 = 16$ triangles. Since the base link is triangle-free and a clique contains at most one new apex, there are no four-vertex cliques. $\square$

The flag-link criterion now proves that Y is nonpositively curved. The three-cubes are essential to this particular certificate: deleting them while retaining its two-skeleton would leave the sixteen link triangles unfilled. This is not a lower bound on the cubical dimension of the group.

5 Identification of the fundamental group

Van Kampen's theorem applied to (4), based at the vertex B of the edge space, gives

$$\pi_1(Y) = \langle x, y, b, s \mid [x, y] = 1, \ bxb^{-1} = y, \ s^{-1}(bx)s = xb^{-1} \rangle. \quad (5)$$

Here s is the interval path at B , oriented from bottom to top; the last relation says $s^{-1}Vs = W$.

Set

$$t = x, \quad a = xy^{-1}, \quad c = b^{-1}s^{-1}. \quad (6)$$

The first relation in (5) gives $tat^{-1} = a$. Since $yb = bx$, the second gives

$$tbt^{-1} = xbx^{-1} = xy^{-1}b = ab.$$

For the final relation, (5) implies $s(xb^{-1})s^{-1} = bx$, and therefore

$$c^{-1}(b^{-1}x)c = sxb^{-1}s^{-1} = bx.$$

Multiplying this equality out yields $xc = bcbx$, or $tct^{-1} = bcb$. Thus the substitutions satisfy (1).

Conversely, starting with (1), define

$$x = t, \quad y = a^{-1}t, \quad s = c^{-1}b^{-1}.$$

The commutation of a and t gives $[x, y] = 1$, and the b relation gives $bxb^{-1} = y$. The c relation is equivalent to $c^{-1}(b^{-1}x)c = bx$, and hence

$$s^{-1}(bx)s = bc(bx)c^{-1}b^{-1} = b(b^{-1}x)b^{-1} = xb^{-1}.$$

The two substitutions are mutually inverse, using $[x, y] = 1$ in recovering y . They therefore give an isomorphism between (5) and (1).

Finally $\tilde{Y}$ is CAT(0) by the universal-cover theorem. A finite unit cube complex has a locally finite universal cover, which is proper in its cubical length metric. The deck action is free and proper, and its quotient is the compact complex Y . This proves Theorem 1.

6 Scope and reproducibility

The construction proves geometric cubulation for the group in (1) itself, without passing to finite index. It does not establish virtual specialness, a minimal cubical dimension, or a general cocompactness theorem for polynomially growing monodromies. In particular it does not transfer to $b \mapsto aba$: the two formulae already have different abelianized mapping-torus groups,

$$G_{ab}^{\text{ab}} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2, \quad G_{aba}^{\text{ab}} \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2.$$

Indeed abelianizing the two mapping-torus presentations gives respectively $a = 0, 2b = 0$ and $2a = 0, 2b = 0$.

The accompanying Python checker reconstructs the links from the oriented edge and square data, checks local injectivity and fullness, verifies the elementary collapses, and enumerates all

$2^{14} - 1$ nonempty subsets of the final link’s vertices. It also checks all actual square relations in the original free-group semidirect-product normal form, including the stable-letter orientation. In total 18,316 exact finite checks agree in ordinary and optimized Python execution. These computations supplement the proof above; they are not a replacement for the flag-link theorem or a formal verification.

References

- [1] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der mathematischen Wissenschaften 319, Springer, 1999.
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