

A Cycle Criterion for CAT(0) Signed Permutation Mapping Tori

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Abstract

Let $F = \langle a, b_1, \dots, b_m \rangle$ and let $\phi(a) = a^\varepsilon$, $\phi(b_i) = a^{p_i} b_{\sigma(i)}^{\delta_i} a^{q_i}$, where $\varepsilon, \delta_i \in \{1, -1\}$, σ is a permutation and p_i, q_i are integers. We give an explicit necessary and sufficient cycle criterion for $F \rtimes_\phi \mathbb{Z}$ to be CAT(0). When the criterion holds, a compact two-dimensional locally CAT(0) graph of spaces realizes the original group, without passing to a power of ϕ . When it fails, no proper semisimple CAT(0) action exists. Positive permutation cycles yield cyclic HNN extensions of a torus or Klein-bottle group; negative cycles yield square-root attachments. For this family, CAT(0) is equivalent for the mapping tori of ϕ and every positive power of ϕ . The unsigned identity-permutation subcase is the criterion of Wu and Ye. The general characterization of CAT(0) free-by-cyclic groups remains open.

1 The family and the criterion

Fix $m \geq 1$, and let

$$\phi(a) = a^\varepsilon, \quad \phi(b_i) = a^{p_i} b_{\sigma(i)}^{\delta_i} a^{q_i} \quad (1 \leq i \leq m). \quad (1)$$

Here σ permutes $\{1, \dots, m\}$, the parameters p_i, q_i are integers, and ε, δ_i are signs. These images form a free basis: multiplication on either side by powers of a , permutation and inversion are Nielsen operations. Thus ϕ is an automorphism. Our convention is

$$G_\phi = F \rtimes_\phi \langle t \rangle, \quad txt^{-1} = \phi(x).$$

A geometric action is a proper cocompact action by isometries. A semisimple isometry is one whose displacement infimum is attained.

For a cycle C of σ , choose its least index i_C , write $l_C = |C|$ and $d_C = \prod_{i \in C} \delta_i$, and define P_C, Q_C by

$$\phi^{l_C}(b_{i_C}) = a^{P_C} b_{i_C}^{d_C} a^{Q_C}. \quad (2)$$

A cycle is called positive or negative according to d_C . All cycle data are integers and are effectively computable.

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Theorem 1. *The group G_ϕ is $CAT(0)$ if and only if the applicable condition below holds.*

1. *If $\varepsilon = 1$, the numbers*

$$\frac{P_C - Q_C}{l_C}$$

are equal over the positive cycles with $P_C + Q_C \neq 0$. The condition is vacuous when there are none.

2. *If $\varepsilon = -1$, every even-length positive cycle satisfies*

$$P_C^2 = Q_C^2.$$

Negative cycles impose no additional condition. In the positive cases G_ϕ has a compact two-dimensional locally $CAT(0)$ model. In the negative cases it has no proper semisimple action on a complete $CAT(0)$ space.

The unsigned case $\varepsilon = \delta_i = 1$, $\sigma = \text{id}$, is exactly Wu–Ye [3, Theorem 1.5]: their equations factor into equality of $p_i - q_i$ on the indices with $p_i + q_i \neq 0$. We do not claim that factorization as a new result. Our cycle criterion allows arbitrary signs and permutations, and the construction acts on the original suspension.

Lyman [4, Theorem A and Corollary B] obtains $CAT(0)$ suspensions of powers for polynomially growing symmetric and palindromic automorphisms, and of the original automorphism in the upper-triangular case. Those results overlap with some positive examples here. Our proof uses the classical Flat Torus Theorem and $CAT(0)$ gluing [2, II.7 and II.11], as do those constructions. It does not classify arbitrary polynomially growing automorphisms. In particular the broad AIM Problem 5.1 [1] is not resolved. Nonabelian twisting words require other arguments; see, for example, Delage [5].

2 Eliminating permutation cycles

Lemma 2. *Put $B = \langle a, t \mid tat^{-1} = a^\varepsilon \rangle$. Then G_ϕ is the fundamental group of a graph of groups with central vertex B and the following edges. For a positive cycle C there is a cyclic HNN edge with stable letter b_C and relation*

$$b_C(a^{Q_C}t^{l_C})b_C^{-1} = a^{-P_C}t^{l_C}. \quad (3)$$

For a negative cycle there is a cyclic leaf vertex $\langle c_C \rangle$ and an amalgamation identifying

$$c_C^2 = a^{Q_C - \varepsilon^{l_C} P_C} t^{2l_C}. \quad (4)$$

Proof. Each original relation $tb_it^{-1} = a^{p_i}b_{\sigma(i)}^{\delta_i}a^{q_i}$ solves uniquely for $b_{\sigma(i)}$ in terms of a, t, b_i . Eliminating all but one generator of a cycle by Tietze transformations leaves precisely

$$t^{l_C}b_Ct^{-l_C} = a^{P_C}b_C^{d_C}a^{Q_C}. \quad (5)$$

The substituted return word is $\phi^{l_C}(b_{i_C})$; no generator from another cycle is used. Every eliminated generator has a recovery expression, so no additional relation has been introduced or lost.

For $d_C = 1$, rearranging (5) gives (3). For $d_C = -1$, put $u = a^{-P_C}t^{l_C}$ and $v = a^{Q_C}t^{l_C}$. Equation (5) becomes $b_Cub_C = v$. The substitution $c_C = b_Cu$ gives

$$c_C^2 = vu = a^{Q_C - \varepsilon^{l_C} P_C} t^{2l_C}.$$

Conversely $b_C = c_Cu^{-1}$ recovers the original relation. Both associated elements in (3), and the right side of (4), have nonzero t -exponent and infinite order in B . Thus all edge maps are injective. $\square$

The base B is a torus group when $\varepsilon = 1$, and a Klein-bottle group when $\varepsilon = -1$. Its embedding can also be seen directly in the original semidirect product: a relation $a^r t^s = 1$ first forces $s = 0$ by projection to $\mathbb{Z}$, then $r = 0$ in the free kernel.

3 Translation-length obstructions

Suppose G_ϕ acts properly by semisimple isometries on a complete CAT(0) space. Translation length is invariant under conjugacy, and $\ell(g^k) = |k|\ell(g)$ for semisimple g . The Flat Torus Theorem applies to the rank-two abelian subgroups below. Properness makes their translation vectors independent.

First let $\varepsilon = 1$, and let u, v be the translation vectors of a, t on a flat for B . For each positive cycle, (3) gives

$$\|Q_C u + l_C v\|^2 = \|-P_C u + l_C v\|^2.$$

Subtracting and factoring gives

$$(P_C + Q_C)((Q_C - P_C)\|u\|^2 + 2l_C\langle u, v \rangle) = 0. \quad (6)$$

Every active positive cycle therefore requires

$$\frac{P_C - Q_C}{l_C} = \frac{2\langle u, v \rangle}{\|u\|^2}.$$

The right side is independent of the cycle.

Now let $\varepsilon = -1$. Use the abelian subgroup $\langle a, t^2 \rangle$, with independent translation vectors u, w . Conjugation by t sends at^2 to $a^{-1}t^2$, so $\|u + w\| = \|-u + w\|$ and consequently $\langle u, w \rangle = 0$. For even l_C , the two elements in (3) have squared translation lengths

$$Q_C^2\|u\|^2 + \frac{l_C^2}{4}\|w\|^2, \quad P_C^2\|u\|^2 + \frac{l_C^2}{4}\|w\|^2.$$

Equality forces $P_C^2 = Q_C^2$. For odd l and any integer r ,

$$(a^r t^l)^2 = t^{2l},$$

so $\ell(a^r t^l) = \ell(t^{2l})/2$, independently of r . There is no extra length condition from an odd positive cycle. This proves necessity in Theorem 1, including its stronger obstruction. A proper cocompact action is semisimple, so the obstruction also excludes geometric actions.

4 Constructing the geometric models

We prove sufficiency directly. If $\varepsilon = 1$, let d be the common active ratio in Theorem 1, or put $d = 0$ if no cycle is active. Give $\mathbb{R}^2$ the positive definite norm

$$N_d(x, y)^2 = (x + dy/2)^2 + y^2.$$

Let a, t translate by $(1, 0), (0, 1)$, respectively. The quotient is a compact flat torus. Equation (6) gives

$$N_d(Q_C, l_C) = N_d(-P_C, l_C)$$

for every positive cycle, including those with $P_C + Q_C = 0$.

If $\varepsilon = -1$, use the Euclidean plane with

$$a(x, y) = (x + 1, y), \quad t(x, y) = (-x, y + 1).$$

The quotient is a compact flat Klein bottle. For even l , the translation length of $a^r t^l$ is $\sqrt{r^2 + l^2}$. For odd l , it is the glide reflection $(x, y) \mapsto (-x + r, y + l)$, with vertical axis $x = r/2$ and translation length l . Thus the two elements in (3) have equal lengths under the assumed criterion.

For each positive cycle attach a Euclidean cylinder of width one, with its boundary circles mapped to closed geodesics representing the two elements of (3). Their lengths agree. For a negative cycle let W_C denote the right side of (4), and put $L_C = \ell(W_C) > 0$. Add a circle of circumference $L_C/2$, and a cylinder of width one and circumference L_C . Attach one boundary to a closed geodesic for W_C in the central surface, and the other by the degree-two local isometry to the added circle. Its fundamental-group relation is exactly $c_C^2 = W_C$.

For clarity, the closed geodesics are not required to be embedded or primitive. Each boundary map lifts to an isometric embedding of a line onto an axis in the universal cover of the corresponding vertex space. The graph-of-spaces gluing construction [2, II.11.18–II.11.21], described for the cyclic HNN case in [4, Section 1.2], therefore applies. The Bass–Serre tree of spaces consists of planes at the central-vertex lifts, lines at the cyclic-vertex lifts, and Euclidean strips along the edges. The strips are glued along complete convex lines. The vertex spaces remain convex and isometrically embedded. One can first add the root amalgamations and then the HNN edges; the specified axis lengths do not change.

The quotient is the finite graph of spaces constructed above and is compact and locally CAT(0). It is a finite piecewise Euclidean complex: the relevant closed geodesics have integral slopes on the torus cover, and finite subdivision accommodates their intersections and covering multiplicities. Its universal cover is complete, proper and CAT(0). The deck action is free, proper and cocompact, and Lemma 2 identifies its group with G_ϕ . This proves Theorem 1.

Corollary 3. *When G_ϕ is CAT(0), its minimum CAT(0) geometric dimension is two.*

Proof. The construction has dimension two. The subgroup $\langle a, t \rangle$, or $\langle a, t^2 \rangle$ in the reversing case, contains a rank-two lattice, whose proper semisimple action requires a two-flat by the Flat Torus Theorem. $\square$

5 Powers and an effective test

The cycle data can be computed without expanding words. Write $\phi^k(b_i) = a^L b_j^s a^R$, starting at $(L, R, j, s) = (0, 0, i, 1)$. At each step, using the old j, s , set

$$(L, R) \mapsto \begin{cases} (\varepsilon L + p_j, \varepsilon R + q_j), & s = 1, \\ (\varepsilon L - q_j, \varepsilon R - p_j), & s = -1, \end{cases} \quad (j, s) \mapsto (\sigma(j), s\delta_j).$$

One traversal of each cycle suffices. Compare active rational ratios by cross-products, rather than pairwise comparisons of all cycles. This uses $O(m)$ integer additions and cross-products. If the input integers have at most B bits, intermediate integers have $O(B + \log m)$ bits. The test is consequently polynomial in the encoded input size.

Lemma 4. *Let M be a positive common multiple of 2 and all $2l_C$. Then ϕ^M fixes a and has the form $\phi^M(b_i) = a^{U_i} b_i a^{V_i}$. The cycle criterion of Theorem 1 is equivalent to the Wu–Ye criterion for G_{ϕ^M} .*

Proof. The latter criterion says that all $U_i - V_i$ with $U_i + V_i \neq 0$ coincide. Suppose first that $\varepsilon = 1$. For a positive cycle the return map has positive sign, and its M/l_C -fold iterate multiplies P_C, Q_C by M/l_C . Changing the chosen generator around that cycle either leaves (P, Q) unchanged or changes it to $(-Q, -P)$. Thus active differences are precisely $M(P_C - Q_C)/l_C$. A negative return map squares to

$$b \mapsto a^{P_C - Q_C} b a^{Q_C - P_C},$$

so all its resulting sums vanish. This proves the equivalence in the first case.

Suppose $\varepsilon = -1$. On an even positive cycle, the return map fixes a , and the resulting differences have magnitude $M(P_C - Q_C)/l_C$. For consecutive representatives the return difference changes sign; the sum changes by a sign as well. If the sum is nonzero, an even positive cycle therefore supplies both opposite differences and forces $P_C = Q_C$. If the sum is zero it is inactive. This is exactly $P_C^2 = Q_C^2$. On an odd positive cycle the return map reverses a and its square fixes the chosen generator, so it is inactive. On a negative cycle the square of the return map sends b to

$$a^{\alpha P_C - Q_C} b a^{\alpha Q_C - P_C}, \quad \alpha = (-1)^{l_C}.$$

For even length its sum is zero; for odd length its difference is zero. These cycles add no further obstruction. $\square$

Corollary 5. *For automorphisms (1) and every integer $k \geq 1$, G_ϕ is CAT(0) if and only if G_{ϕ^k} is CAT(0).*

Proof. A geometric action restricts to the finite-index subgroup G_{ϕ^k} . Conversely, if that subgroup is CAT(0), choose M in Lemma 4 divisible also by k . Its finite-index subgroup G_{ϕ^M} is CAT(0). Wu–Ye’s theorem and Lemma 4 give our cycle criterion, and the direct construction gives a geometric action of G_ϕ . $\square$

This argument does not assert that arbitrary finite extensions of CAT(0) groups are CAT(0).

6 Examples and verification

If $\varepsilon = 1$ and

$$\phi(b) = ab, \quad \phi(c) = d, \quad \phi(d) = a^2c,$$

the active cycle ratios are 1 and $2/2 = 1$. Hence the group is CAT(0). Omitting the cycle-length divisor would incorrectly reject this example.

If instead $\varepsilon = -1$ and

$$\phi(b) = c, \quad \phi(c) = ab,$$

there is a positive cycle of length two with $(P, Q) = (1, 0)$. The group has no proper semisimple CAT(0) action. A single cycle need not be harmless when the distinguished generator is reversed.

If $\varepsilon = 1$ and every b_i maps to $a^{p_i} b_i^{-1} a^{q_i}$, every cycle is negative. All these groups are CAT(0), for arbitrary integers p_i, q_i . The unsigned criterion is not applicable to these signed data.

The accompanying exact-arithmetic program compares the recurrence with literal freely reduced substitutions, checks the HNN and square-root identities, and compares the criterion with the unsigned criterion on a positive power. It covers 10,423 parameter choices and 254,789 individual checks, with identical results in normal and optimized execution. These are finite regression checks, not formal verification or a replacement for the proof above.

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