# Forcing Absoluteness of Minimal and Maximal $C^*$-Tensor Products

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**Abstract**

We answer an AIM question about whether the minimal or maximal $C^*$-tensor product commutes with forcing. We use the standard convention that a ground-model $C^*$-algebra is replaced in the extension by the metric completion of its old normed $*$-algebra. Under this convention, both $\otimes_{\min}$ and $\otimes_{\max}$ commute canonically with every set-forcing extension, for arbitrary ground-model factors; no separability, density-character, or cardinal-preservation hypothesis is needed. The minimal case follows by extension of faithful spatial representations. For the maximal case, any putative larger new norm would give finitely satisfiable old semialgebraic $C^*$-seminorm constraints. Quantifier elimination for real closed fields and compactness inside the ground model then produce an old $C^*$-seminorm exceeding the old universal norm, a contradiction.

## 1 Introduction and convention

AIM Problem 3.3 from the 2012 workshop *Set theory and $C^*$-algebras* asks: "Does $\otimes_{\min}$ or $\otimes_{\max}$ commute with forcing?" [1]. The one-line source does not specify a comparison convention. We adopt the standard convention used in forcing work on $C^*$-algebras: in the extension, complete the ground-model normed $*$-algebra in the extension's metric universe [3, Section 2.2]. This differs in general from recomputing a named construction from scratch.

Let $M \subseteq N = M[G]$ be transitive models of ZFC, with $N$ a set-forcing extension of $M$. If $A \in M$ is a $C^*$-algebra, write $A^N$ for the completion in $N$ of its ground-model normed $*$-algebra. If $C$ is a ground-model normed algebra already completed in $M$, write $\widehat{C}^{,N}$ for the same extension completion.

**Theorem 1.1.** *Let $A, B \in M$ be $C^*$-algebras, not necessarily unital or separable, and put*

$$D = A^M \odot_{\mathbb{C}^M} B^M.$$

*For $\alpha \in \{\min, \max\}$, the identity on $D$ extends uniquely to a canonical isometric $*$-isomorphism*

$$(\widehat{A \otimes_\alpha^M B})^{M,N} \cong A^N \otimes_\alpha^N B^N.$$

*No preservation of cardinals or density characters is required.*

The standard spatial and universal descriptions of the two norms are recalled in [2, 4]. The only set-theoretic input beyond forcing terminology is ordinary first-order absoluteness for finite semialgebraic feasibility.

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## 2 Preliminary lemmas

**Lemma 2.1** (old bounded operators retain their norm)**.** *Let $X, Y \in M$ be normed spaces and let $T : X \to Y$ be bounded in $M$. If $T^N : X^N \to Y^N$ is its continuous extension, then*

$$\|T^N\|^N = \|T\|^M.$$

*Proof.* Put $c = \|T\|^M$. The old inequality $\|Tx\| \le c\|x\|$ remains valid and extends by density, so $\|T^N\| \le c$. For every positive old rational $\varepsilon$, an old unit vector satisfies $\|Tx\| > c - \varepsilon$. The same vector is present in $N$, giving the reverse inequality as $\varepsilon \downarrow 0$. □

**Lemma 2.2** (common density)**.** *For either $C^*$-cross norm $\alpha \in \{\min, \max\}$, the old algebraic tensor $D$ is dense in both algebras in Theorem 1.1, when viewed in $N$.*

*Proof.* Density on the left is definitional. On the right, write an algebraic tensor as $\sum_{j=1}^{n} a_j \otimes b_j$ with $a_j \in A^N$, $b_j \in B^N$, and choose old approximants $a_{j,k} \to a_j$, $b_{j,k} \to b_j$. Every $C^*$-cross norm satisfies

$$\|a \otimes b - a_0 \otimes b_0\|_\alpha \le \|a - a_0\| \, \|b\| + \|a_0\| \, \|b - b_0\|.$$

The corresponding old finite sums converge. This also handles complex scalars newly present in the forcing extension. □

**Lemma 2.3** (finite real feasibility descends)**.** *Every finite system of polynomial equalities and weak or strict inequalities with coefficients in the real field $\mathbb{R}^M$ that has a solution in $\mathbb{R}^N$ already has a solution in $\mathbb{R}^M$.*

*Proof.* Both fields are real closed. Quantifier elimination for real closed ordered fields makes every embedding of real closed ordered fields elementary [5, Theorem 4.3.2]. Existence of a solution is an existential first-order formula with old parameters. □

The next lemma is the key nonseparable maximal-norm step.

**Lemma 2.4** (new seminorms are bounded by the old universal norm)**.** *Let $E \in M$ be a set-sized complex $*$-algebra. Suppose that in $N$ a map $q : E \to \mathbb{R}^N$ is a $C^*$-seminorm on the old algebra, homogeneous for old complex scalars. Assume that for every $x \in E$ there is $B_x \in \mathbb{R}^M$ with $q(x) \le B_x$. Whenever the old universal norm is finite,*

$$q(x) \le \|x\|^M_{\mathrm{univ}} \qquad (x \in E).$$

*Proof.* Fix $z \in E$ and suppose $q(z) > c := \|z\|^M_{\mathrm{univ}}$. Choose an old rational $r$ with $c < r < q(z)$, choose old positive coordinate bounds $B_x$, and form inside $M$ the compact product

$$K = \prod_{x \in E} [0, B_x].$$

For $p \in K$, impose, for all applicable old $x, y$ and $\lambda \in \mathbb{C}^M$, the closed cylinder conditions

$$\begin{aligned} p(0) &= 0, & p(\lambda x) &= |\lambda| p(x), \\ p(x+y) &\le p(x) + p(y), & p(xy) &\le p(x)p(y), \\ p(x^*) &= p(x), & p(x^* x) &= p(x)^2, \end{aligned}$$

together with $p(z) \geq r$.

Every finite subfamily involves finitely many real coordinates, and the values of $q$ solve it in $\mathbb{R}^N$. Lemma 2.3 descends a solution to $\mathbb{R}^M$. Thus $M$ sees the finite intersection property for these closed subsets of $K$. Tychonoff compactness, applied inside $M$, produces a ground-model $C^*$-seminorm $p$ satisfying all conditions and $p(z) \geq r$.

Complete the quotient of $E$ by the kernel of $p$ and faithfully represent the resulting $C^*$-algebra, all inside $M$. By definition of the universal norm, $p(x) \leq \|x\|^M_{\mathrm{univ}}$, contradicting $p(z) \geq r > c$. □

**Remark 2.5.** *The compactness argument is internal to $M$. It does not claim that a ground-model compact product remains externally compact after forcing.*

# 3 The minimal tensor norm

Choose in $M$ faithful representations

$$\pi : A \to \mathcal{B}(H), \qquad \sigma : B \to \mathcal{B}(K).$$

Completing the Hilbert spaces and extending old bounded operators in $N$, Lemma 2.1 gives faithful isometric representations

$$\pi^N : A^N \to \mathcal{B}(H^N), \qquad \sigma^N : B^N \to \mathcal{B}(K^N).$$

The completion in $N$ of the old Hilbert tensor product is canonically $H^N \otimes K^N$, because old algebraic simple tensors are dense on both sides. For $d = \sum_j a_j \otimes b_j \in D$, the new spatial operator is the continuous extension of the old one. Lemma 2.1 therefore gives

$$\|d\|^N_{\min} = \left\| \sum_j \pi^N(a_j) \otimes \sigma^N(b_j) \right\|^N = \left\| \sum_j \pi(a_j) \otimes \sigma(b_j) \right\|^M = \|d\|^M_{\min}.$$

Equality on $D$, followed by Lemma 2.2, proves Theorem 1.1 for $\alpha = \min$.

# 4 The maximal tensor norm

For each $d \in D$, fix in $M$ a representation $d = \sum_{j=1}^{n(d)} a_{d,j} \otimes b_{d,j}$ and an old positive integer

$$B_d > \sum_j \|a_{d,j}\| \, \|b_{d,j}\|.$$

Define

$$q(d) = \|d\|_{A^N \otimes^N_{\max} B^N}.$$

Restriction to the old algebra makes $q$ a $C^*$-seminorm on $D$; no injectivity of the old algebraic map is assumed. The projective estimate gives $q(d) < B_d$. Lemma 2.4 applies, and since the universal $C^*$-norm on $D$ is the old maximal tensor norm,

$$q(d) \leq \|d\|^M_{\max}. \tag{1}$$

For the reverse inequality, take a ground-model commuting pair

$$\rho_A : A \to \mathcal{B}(L), \qquad \rho_B : B \to \mathcal{B}(L).$$

Both maps extend continuously in $N$ to $A^N$ and $B^N$ on $L^N$. Commutation persists by density, and Lemma 2.1 preserves the norm of every old finite-tensor operator. Hence every old near-witness for the maximal norm remains a witness in $N$, so

$$\|d\|^M_{\max} \le q(d). \tag{2}$$

Equations (1) and (2) give equality on $D$; Lemma 2.2 completes the proof for $\alpha = \max$.

The nonunital case is unchanged. The maximal norm is the largest $C^*$-norm on the algebraic tensor product, equivalently the supremum from commuting nondegenerate representations after the standard unitization or approximate-unit argument. Lemma 2.4 uses only the largest norm property and never inserts a unit into $D$.

## 5 Canonicity and scope

On both sides, the map is the unique continuous extension of $a \otimes b \mapsto a \otimes b$ for old $a, b$. Equality of norms makes it isometric, and Lemma 2.2 makes its range dense; an isometric range is closed, hence the map is onto.

No countability or cardinal-preservation step occurs. The result concerns the extension completion of the old metric algebras. It does not assert that this completion agrees with every named $C^*$-algebraic construction recomputed from scratch in the extension. This convention is essential because the AIM source itself does not specify the comparison map.


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