

Finite-Frequency Rigidity for Maximal Perimeter Sections

Alper Ferudun*

October 15, 2026

Abstract

We prove a quantitative nonlinear symmetry theorem for convex bodies in $\mathbb{R}^3$ whose radial functions are close to one in C^2 and whose odd parts have bounded spherical harmonic degree. The even part is unrestricted. An explicit bound controls the odd radial part by the derivatives at zero of the parallel-section perimeters. For an arbitrary odd part, the same estimate has an additional C^2 high-frequency-tail term. The proof combines the classical equatorial derivative transform with a finite-rank inverse and an explicit nonlinear remainder bound. We also prove vanishing of all odd parameter jets for smooth solution families through the ball, and symmetry of real-analytic families. An exactly evaluated ellipsoid example identifies a missing term in a previously printed directional first-variation formula, without affecting the classical infinitesimal theorem at the ball. The neighborhood depends on the harmonic cutoff; the general maximal-perimeter symmetry conjecture is not resolved.

1 The quantitative theorem

Let $K \subset \mathbb{R}^3$ be a convex body with $0 \in \text{int } K$ and positive radial function $\rho \in C^2(\mathbb{S}^2)$. For $u \in \mathbb{S}^2$, put

$$P_u(t) = \text{Per}(K \cap (tu + u^\perp)), \quad F(\rho)(u) = P'_u(0).$$

Perimeter is measured in the section plane. The derivative exists under these assumptions, as established in Section 2. On the round unit sphere use the norm

$$\|f\|_{C^2} = \max \left\{ \|f\|_\infty, \|\nabla f\|_\infty, \sup_{\omega \in \mathbb{S}^2} \|\nabla^2 f(\omega)\|_{\text{op}} \right\}. \quad (1)$$

Write $f_e(\omega) = (f(\omega) + f(-\omega))/2$ and $f_o(\omega) = (f(\omega) - f(-\omega))/2$. For an integer $L \geq 1$, let V_L be the span of real spherical harmonics of odd degrees at most L , and let Π_L be the $L^2(\mathbb{S}^2)$ orthogonal projection onto V_L . Define

$$B_L = \sum_{\substack{1 \leq \ell \leq L \\ \ell \text{ odd}}} \ell(\ell+1)(2\ell+1)^2. \quad (2)$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

Theorem 1.1. *If $\rho_o \in V_L$ and*

$$\|\rho - 1\|_{C^2} \leq \frac{1}{204B_L}, \quad (3)$$

then

$$\|\rho_o\|_{C^2} \leq \frac{B_L}{\pi} \sup_{u \in \mathbb{S}^2} |P'_u(0)|. \quad (4)$$

In particular, if every central planar section has maximum perimeter among its parallel sections, then $K = -K$. Without the assumption $\rho_o \in V_L$, condition (3) still implies

$$\|\rho_o\|_{C^2} \leq \frac{B_L}{\pi} \|F(\rho)\|_\infty + 2\|(I - \Pi_L)\rho_o\|_{C^2}. \quad (5)$$

The even part of ρ need not have finite harmonic degree. For example, $B_1 = 18$, $B_3 = 606$ and $B_5 = 4236$. The constants are deliberately conservative. This is a derivative-based estimate, not an estimate in terms of the central-to-maximal perimeter deficit. In (5), the tail is measured in C^2 , not merely in L^2 .

Makai, Martini and Ódor established the volume-section symmetry theorem and the kernel of the equatorial derivative transform [1]. Their Problem 3.10 asks for a quermassintegral extension. Makai and Martini proved that the first radial variation of a sufficiently smooth family through the ball is even under the corresponding maximality assumption [2]. Stephen proved the full perimeter implication for convex polytopes [3]. We use the classical transform and its kernel, but the present conclusion is a nonlinear estimate for a single body with an odd harmonic cutoff.

The related AIM question [4] is recorded as `AIM-FUNCTIONAL_ANALYSIS-0012` in the frozen `ulamai/UnsolvedMath v1.6.0` dataset. For a hyperplane section in $\mathbb{R}^n$, the intended boundary measure has dimension $n - 2$ and is taken relative to that hyperplane; this is explicit in the cited primary papers. The frozen displayed $n - 1$ subscript is not used as a different perimeter functional. The inherited research already contains the dimension repair and a translated-ellipsoid deficit calculation. Neither is claimed as a new result here.

2 Exact first variation

Fix $u \in \mathbb{S}^2$, parametrize its equator by a unit-speed circle $\eta(\theta)$, and use geographic latitude ψ :

$$\omega(\theta, \psi) = \eta(\theta) \cos \psi + u \sin \psi.$$

All quantities in the next display are restricted to $\psi = 0$:

$$r = \rho, \quad p = \rho_\theta, \quad q = \rho_{\theta\theta}, \quad b = \rho_\psi, \quad R = (r^2 + p^2)^{1/2}.$$

Let $s(\theta, t)$ be the radial function of the section, measured from tu inside its plane. The identities

$$s = \rho(\theta, \psi) \cos \psi, \quad t = \rho(\theta, \psi) \sin \psi$$

and the implicit function theorem give, at $t = 0$,

$$s = r, \quad s_\theta = p, \quad s_t = \frac{b}{r}, \quad s_{\theta t} = \frac{b_\theta}{r} - \frac{bp}{r^2}. \quad (6)$$

The last derivative is taken at fixed section height. Positivity of ρ and compactness make these local formulas uniform around the equator for sufficiently small t .

Lemma 2.1. *The exact first variation is*

$$F(\rho)(u) = \int_0^{2\pi} \frac{b + pb_\theta/r - p^2b/r^2}{R} d\theta \quad (7)$$

$$= \int_0^{2\pi} A(r, p, q)b d\theta, \quad A(r, p, q) = \frac{r^2 + 2p^2 - rq}{(r^2 + p^2)^{3/2}}. \quad (8)$$

The expression in (8) extends to a smooth map from positive $C^2(\mathbb{S}^2)$ functions to $C^0(\mathbb{S}^2)$.

Proof. Differentiate $P_u(t) = \int_0^{2\pi} (s^2 + s_\theta^2)^{1/2} d\theta$ and use (6). This gives (7). Integration by parts on the closed equator uses

$$\frac{r}{R} - \left(\frac{p}{R}\right)_\theta = \frac{r(r^2 + 2p^2 - rq)}{R^3}.$$

Multiplying this Euler derivative by $s_t = b/r$ yields (8). Its coefficient A is the curvature of the radial plane curve. The formula uses at most two spatial derivatives; its denominator stays uniformly positive on a sufficiently small C^2 neighborhood of any fixed positive function. Differentiation of this jet expression and equatorial integration are bounded operations, which proves the asserted smoothness. The expression is in fact real analytic in ρ . $\square$

For every positive even function e , antipodal pairing gives $F(e) = 0$. Equivalently, reflection preserves its radial body and identifies the sections at t and $-t$. Convexity is not needed for this local identity. At the ball, Lemma 2.1 gives

$$DF(1)[f] = Tf, \quad Tf(u) = \int_0^{2\pi} f_\psi(\eta(\theta), 0) d\theta. \quad (9)$$

Remark 2.2. For a convex body in $\mathbb{R}^3$, the parallel-section perimeter profile is concave on the interval of nondegenerate sections. Indeed, translate all sections to $u^\perp$. Convexity gives $(1 - a)C_s + aC_t \subseteq C_{(1-a)s+at}$ for $0 \leq a \leq 1$. Planar perimeter is monotone and Minkowski additive, for instance by its support-function integral, so

$$P_u((1 - a)s + at) \geq (1 - a)P_u(s) + aP_u(t).$$

Thus differentiable stationarity at an interior point is equivalent to global maximality. Inward homothetic containment bounds any supporting endpoint section by the one-sided interior limit. Only the implication from central maximality to $F(\rho) = 0$ is needed below.

3 A finite spectral inverse

Let $Y_{\ell j}$, $1 \leq j \leq 2\ell + 1$, be a real orthonormal harmonic basis with respect to surface measure $d\sigma$ of mass 4π . Let P_ℓ be the Legendre polynomial normalized by $P_\ell(1) = 1$. The classical multiplier identity is

$$TY_{\ell j} = \lambda_\ell Y_{\ell j}, \quad \lambda_\ell = 2\pi P'_\ell(0). \quad (10)$$

Indeed, averaging $Y_{\ell j}$ over the latitude $\omega \cdot u = t$ with angular measure gives $2\pi P_\ell(t)Y_{\ell j}(u)$. Differentiate at $t = 0$, where the t and latitude derivatives agree. For $\ell = 2m + 1$,

$$|P'_{2m+1}(0)| = \frac{(2m+1)!}{2^{2m}(m!)^2} \geq 1. \quad (11)$$

The first value is one and the successive ratio is $(2m+3)/(2m+2) > 1$. Even eigenvalues are zero. In particular, the kernel of T on C^1 consists exactly of even functions [1, Theorem 3.8]. This also follows directly from (10): latitude averaging is self-adjoint, hence so is its derivative when paired with harmonics, and every odd harmonic coefficient of a kernel element vanishes.

Define

$$Q_L g = \sum_{\substack{1 \leq \ell \leq L \\ \ell \text{ odd}}} \sum_{j=1}^{2\ell+1} \frac{\int_{\mathbb{S}^2} g Y_{\ell j} d\sigma}{\lambda_\ell} Y_{\ell j}.$$

Then $Q_L T o = \Pi_L o$ for odd $o \in C^2(\mathbb{S}^2)$.

Lemma 3.1. *For $g \in C^0(\mathbb{S}^2)$,*

$$\|Q_L g\|_{C^2} \leq \frac{B_L}{2\pi} \|g\|_\infty. \quad (12)$$

Proof. Put $a_\ell = \ell(\ell+1)$ and $d_\ell = 2\ell+1$. The addition theorem and its gradient version give

$$\sum_j Y_{\ell j}^2 = \frac{d_\ell}{4\pi}, \quad \sum_j |\nabla Y_{\ell j}|^2 = \frac{a_\ell d_\ell}{4\pi}.$$

The sum of squared Frobenius norms of their Hessians is rotation invariant. Its integral equals $d_\ell a_\ell (a_\ell - 1)$ by the integrated Bochner identity on $\mathbb{S}^2$, whose Ricci curvature is one. Consequently,

$$\sum_j \|\nabla^2 Y_{\ell j}\|_F^2 = \frac{d_\ell a_\ell (a_\ell - 1)}{4\pi}.$$

For $\ell \geq 1$, these identities imply $\|Y_{\ell j}\|_{C^2} \leq a_\ell \sqrt{d_\ell/(4\pi)}$. Also $\|Y_{\ell j}\|_{L^1} \leq \sqrt{4\pi}$ by Cauchy–Schwarz. Together with (11), the contribution of degree ℓ to the operator norm is at most $a_\ell d_\ell^{3/2}/(2\pi) \leq a_\ell d_\ell^2/(2\pi)$. Summing gives (12). $\square$

4 The nonlinear estimate and proof of rigidity

Lemma 4.1. *If $\|\rho - 1\|_{C^2} \leq \delta \leq 1/4$, then*

$$\|DF(\rho) - T\|_{C^2 \rightarrow C^0} \leq 204\pi\delta. \quad (13)$$

Proof. In the notation of (8), $3/4 \leq r \leq 5/4$ and $|p|, |q|, |b| \leq \delta$. Set $N = r^2 + 2p^2 - rq$; then $|N| \leq 2$ and $R \geq 3/4$. The partial derivatives of $A = N/R^3$ are

$$A_r = \frac{2r - q}{R^3} - \frac{3rN}{R^5}, \quad A_p = \frac{4p}{R^3} - \frac{3pN}{R^5}, \quad A_q = -\frac{r}{R^3}.$$

Their absolute values are bounded, respectively, by

$$\frac{3088}{81} < 39, \quad \frac{704}{81} < 9, \quad \frac{80}{27} < 3.$$

For example the first bound is $(11/4)(64/27) + (15/2)(1024/243)$, and the second is $64/27 + (3/2)(1024/243)$. The segment from $(1, 0, 0)$ to (r, p, q) stays in this box. Since $A(1, 0, 0) = 1$, $|A - 1| \leq 51\delta$.

Differentiating (8) in direction f gives

$$DF(\rho)[f](u) = \int_0^{2\pi} [Af_\psi + (A_rf + A_pf_\theta + A_qf_{\theta\theta})b] d\theta.$$

The second θ derivative along the great circle is the intrinsic Hessian applied to its unit tangent twice. Each of the two terms in $(DF(\rho) - T)f$ is bounded by $51\delta\|f\|_{C^2}$. Integration over length 2π proves (13). $\square$

Proof of Theorem 1.1. Write $\rho = e + o$ for its even and odd parts. For $0 \leq s \leq 1$,

$$e + so = \frac{1+s}{2}\rho + \frac{1-s}{2}\rho(-\cdot).$$

The antipodal map preserves (1), so this segment remains within $\delta = 1/(204B_L)$ of one. Since $B_L \geq 18$, Lemma 4.1 applies. The identity $F(e) = 0$ and the fundamental theorem of calculus give

$$Q_L F(\rho) = \Pi_L o + \int_0^1 Q_L [DF(e + so) - T] o ds. \quad (14)$$

By Lemmas 3.1 and 4.1, the remainder has C^2 norm at most

$$\frac{B_L}{2\pi} 204\pi\delta\|o\|_{C^2} = \frac{1}{2}\|o\|_{C^2}.$$

If $o \in V_L$, then $\Pi_L o = o$, and $\|o\|_{C^2} \leq 2\|Q_L F(\rho)\|_{C^2} \leq (B_L/\pi)\|F(\rho)\|_\infty$. Central maximality implies $F(\rho) = 0$, so $o = 0$ and $K = -K$.

For an arbitrary odd part put $h = (I - \Pi_L)o$. Using $o = \Pi_L o + h$ in (14) yields

$$\|o\|_{C^2} \leq \|Q_L F(\rho)\|_{C^2} + \frac{1}{2}\|o\|_{C^2} + \|h\|_{C^2}.$$

Rearrangement proves (5). $\square$

5 Parameter jets and analytic families

Proposition 5.1. *Let $k \geq 1$ and let $\lambda \mapsto \rho_\lambda$ be a C^k map from a neighborhood of zero into $C^2(\mathbb{S}^2)$, with $\rho_0 = 1$. Suppose every ρ_λ is the positive radial function of a convex body with all central planar sections of maximal perimeter. Then every parameter derivative through order k at zero is even. If the family is real analytic on a connected open interval containing zero, it is even throughout that interval.*

Proof. Let e_λ be the even part of ρ_λ . Induct on the first possibly nonzero odd jet. After the preceding odd jets have vanished, Taylor's formula in C^2 gives

$$\rho_\lambda = e_\lambda + \lambda^m o_m + o(|\lambda|^m),$$

where o_m is odd. Since $F(e_\lambda) = F(\rho_\lambda) = 0$, smoothness and $e_\lambda \rightarrow 1$ imply

$$0 = F(\rho_\lambda) - F(e_\lambda) = \lambda^m T o_m + o(|\lambda|^m) \quad \text{in } C^0.$$

The classical kernel statement following (11) implies $o_m = 0$. This proves the induction. For an analytic family the C^2 Taylor series converges, so its odd part vanishes near zero. Apply the scalar identity theorem to the odd part at each fixed $\omega \in \mathbb{S}^2$ to extend this conclusion along the interval. $\square$

This proposition requires differentiability as a map into C^2 . A C^∞ odd part can be flat in the parameter; the jet conclusion alone does not rule that out. Moreover, an arbitrary body is not assumed to lie on an analytic path of solutions through the ball.

6 An ellipsoid test of a printed directional formula

In equation (15) of [2], physical page 8 of arXiv v1 and page 11 of the author PDF, directional stationarity is expressed using

$$J(\rho, u) = \int_0^{2\pi} \frac{b + pb_\theta/r}{R} d\theta.$$

Comparison with (7) gives

$$J(\rho, u) = F(\rho)(u) + D(\rho, u), \quad D(\rho, u) = \int_0^{2\pi} \frac{p^2 b}{r^2 R} d\theta. \quad (15)$$

Near the ball $D = O(\|\rho - 1\|_{C^2}^3)$, so the missing term does not change the source's infinitesimal theorem at the ball. In the plane, W_1 is a positive constant multiple of perimeter, which has the same zero-derivative condition.

Proposition 6.1. *For $0 < a < 1$, let*

$$K_a = \{(x, y, z) \in \mathbb{R}^3 : (x - a - z)^2 + y^2 + z^2 \leq 1\}.$$

Its radial function ρ_a satisfies $F(\rho_a)(e_3) = 0$, whereas

$$J(\rho_a, e_3) = \frac{\pi a}{1 - a^2} \left[\frac{2(1 - \sqrt{1 - a^2})}{a^2} - (1 + a^2) \right]. \quad (16)$$

In particular, $J(\rho_{1/2}, e_3) = \pi(27 - 16\sqrt{3})/6 < 0$.

Proof. The ellipsoid is an invertible affine image of the unit ball and contains zero in its interior. Its section at $z = t$ is the circle centered at $(a + t, 0, t)$ of radius $\sqrt{1 - t^2}$, for $|t| < 1$. Thus $P_{e_3}(t) = 2\pi\sqrt{1 - t^2}$, giving the asserted zero derivative. At $t = 0$ the origin-centered polar radius and its height derivative are

$$r(\theta) = a \cos \theta + \sqrt{1 - a^2 \sin^2 \theta}, \quad v = s_t(\theta, 0) = \cos \theta - \frac{a \sin^2 \theta}{\sqrt{1 - a^2 \sin^2 \theta}}.$$

Equation (6) gives $b = rv$. Parametrize the section circle instead by $(a + \cos \varphi, \sin \varphi, 0)$, and write $c = \cos \varphi$. Then

$$\begin{aligned} r^2 &= 1 + a^2 + 2ac, & d\theta &= \frac{1 + ac}{r^2} d\varphi, \\ p &= -\frac{ra \sin \varphi}{1 + ac}, & R &= \frac{r^2}{1 + ac}, & v &= \frac{rc}{1 + ac}. \end{aligned}$$

All denominators are positive. Substitution into (15) therefore gives

$$D(a) = a^2 \int_0^{2\pi} \frac{\sin^2 \varphi \cos \varphi}{(1 + a \cos \varphi)(1 + a^2 + 2a \cos \varphi)} d\varphi.$$

Use the exact partial fraction identity

$$\frac{c}{(1+ac)(1+a^2+2ac)} = \frac{1}{a(1-a^2)} \left(\frac{1}{1+ac} - \frac{1+a^2}{1+a^2+2ac} \right)$$

and, for $A > |B|$ and $B \neq 0$, the integral

$$\int_0^{2\pi} \frac{\sin^2 \varphi}{A + B \cos \varphi} d\varphi = \frac{2\pi(A - \sqrt{A^2 - B^2})}{B^2}.$$

The last formula follows by polynomial division in $\cos \varphi$ and the half-angle substitution for the reciprocal linear denominator. These identities yield (16), since $F(\rho_a)(e_3) = 0$. At $a = 1/2$ it reduces to the stated expression, whose strict sign follows from $27^2 = 729 < 768 = (16\sqrt{3})^2$. $\square$

The example corrects the printed expression as a general test of *directional* stationarity. It is not a counterexample to the all-direction symmetry conjecture: K_a has center $(a, 0, 0)$ and does not satisfy central maximality in every direction. The comparison above is with the identified author and arXiv versions, not a separately inspected publisher typesetting.

7 Scope and reproducibility

The argument proves the stated finite-frequency theorem, the C^2 -tail bound and the parameter-family proposition. It does not provide a neighborhood independent of L , a bound from the perimeter deficit, or a resolution of the general AIM question. The transform kernel, infinitesimal theorem and polytope theorem are credited to the sources above. A bounded literature search is not a certificate of absolute priority.

The source archive includes the analytic proof, source comparison and `verify_rigidity.py`. Its 690 explicit checks pass in normal and optimized Python with identical output. Rational-arithmetic checks cover derivative constants, Legendre coefficients, contraction factors and a partial fraction identity. The latitude multiplier and ellipsoid quadratures are supplementary floating-point regressions, not exact certificates. The proof does not depend on the finite tests and has not been formalized in a proof assistant or independently peer reviewed.

AI assistance was used for mathematical derivation, source comparison, verification code and manuscript preparation. The author is responsible for all claims.

References

- [1] E. Makai Jr., H. Martini and T. Ódor, *Maximal sections and centrally symmetric bodies*, *Mathematika* **47** (2000), 19–30. doi:10.1112/S0025579300015680.
- [2] E. Makai Jr. and H. Martini, *Centrally symmetric convex bodies and sections having maximal quermassintegrals*, *Studia Scientiarum Mathematicarum Hungarica* **49** (2012), no. 2. doi:10.1556/SScMath.49.2012.2.1197. Version compared: arXiv:1507.01467v1.
- [3] M. Stephen, *Maximal perimeters of polytope sections and origin-symmetry*, *Discrete & Computational Geometry* **68** (2022), 349–371. doi:10.1007/s00454-022-00401-z. Preprint: arXiv:1803.01506v1.
- [4] American Institute of Mathematics, *Symmetry and convexity in geometric inequalities*, AIM Problem Lists, section 2, item 2.2. <http://aimpl.org/symconvgeomineq/2/>. Wording preserved in `ulamai/UnsolvedMath`, version 1.6.0, record AIM-FUNCTIONAL_ANALYSIS-0012; the intended perimeter formulation is corroborated by [1, 2, 3].