Ramification Portraits of Rigid Lattès Maps

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Abstract

We classify the abstract weighted ramification portraits of rigid complex Lattès maps. For a Lattès map induced by an affine torus endomorphism of degree d, the complete portrait is determined by its action ϕ on the finite branch-value set and by a uniform fiber formula. Reducing the affine map on the four possible Euclidean orbifolds yields nine affine functional graphs for signature $(2,2,2,2)$, four graphs for $(3,3,3)$, four Gaussian parity cases for $(2,4,4)$, and four Eisenstein divisibility cases for $(2,3,6)$. We give the critical-leaf multiplicities in every case and prove realizability. We also answer the overlap question from AIM Problem 6.4: every flexible Lattès portrait, in every possible square degree, occurs for a rigid Lattès map of the same degree. The result classifies weighted directed graphs; it does not classify maps up to Möbius conjugacy or retain cross-ratios and multiplier data.

1 Introduction

The AIM workshop on postcritically finite maps asked which ramification portraits occur for rigid Lattès maps and, in particular, whether any are portraits of flexible Lattès maps [1]. We use the weighted dynamic-portrait convention: the vertices are the union of the critical and postcritical sets, every vertex has one outgoing arrow, and that arrow is weighted by the local degree. This is the convention used for NET maps by Floyd, Parry, and Pilgrim [3]; see also the general portrait formalism of Doyle and Silverman [2].

Milnor's conformal description [4] writes a Lattès map as the quotient of an affine torus map $L(z) = \alpha z + \beta$ on E, with quotient $\pi : E \to \mathbb{P}^1$ and commutative relation

$$f \circ \pi = \pi \circ L. \tag{1.1}$$

where $E = \mathbb{C}/\Lambda$, the map π is the quotient by a cyclic rotation group G_n, and $n \in \{2, 3, 4, 6\}$. Put $d = \deg f = |\alpha|^2 > 1$. The map is flexible precisely when $n = 2$ and $\alpha \in \mathbb{Z}$; the remaining cases are rigid.

The classification below is finite because all critical points map directly to the branch-value set of π. It consists of one universal formula and four residue calculations. The construction in Section 7 then gives a uniform affirmative answer to the flexible-overlap question.

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2 A universal portrait formula

Let $B \subset E/G_n$ be the branch-value set of π. It is the postcritical set of f. For $v = \pi(z) \in B$, write

$$e(v) = |\operatorname{Stab}_{G_n}(z)|.$$

The multisets of orbifold weights are, respectively,

$$(2,2,2,2), \qquad (3,3,3), \qquad (2,4,4), \qquad (2,3,6). \tag{2.1}$$

The map L sends branch orbits to branch orbits; let $\phi : B \to B$ be the induced map.

Proposition 2.1 (Fiber formula)**.** *The complete weighted portrait is obtained from (B, e, ϕ) as follows. For every $u \in B$, draw*

$$u \longrightarrow \phi(u) \quad \textit{with weight} \quad \frac{e(\phi(u))}{e(u)}. \tag{2.2}$$

For each $v \in B$, attach

$$\kappa_v = \frac{1}{e(v)} \left(d - \sum_{\substack{u \in B \\ \phi(u)=v}} \frac{e(v)}{e(u)} \right) \tag{2.3}$$

new critical leaves, each with one arrow of weight $e(v)$ to v. There are no other portrait vertices or arrows.

Proof. Taking local degrees in $f\pi = \pi L$ and using that the affine isogeny is unramified gives

$$\deg_{\pi(z)} f = \frac{e(\pi(Lz))}{e(\pi(z))}. \tag{2.4}$$

The stabilizer of z is contained in the stabilizer of Lz, so the ratio is an integer. Every critical point therefore maps to a branch value. A nonbranch preimage of v has local degree $e(v)$, whereas a branch preimage u has local degree $e(v)/e(u)$. Summing local degrees in the fiber above v gives (2.3). Since the branch set is the postcritical set, this accounts for every vertex in the weighted portrait. □

Thus it remains only to classify the finite weighted branch skeleton.

3 The four-point signature (2,2,2,2)

Here $B = E[2] \cong \mathbb{F}_2^2$, all four weights are 2, and

$$\phi(x) = Ax + b, \qquad A \in M_2(\mathbb{F}_2), \quad b \in \mathbb{F}_2^2. \tag{3.1}$$

If $r_v = |\phi^{-1}(v)|$, then (2.3) becomes

$$\kappa_v = \frac{d - r_v}{2}. \tag{3.2}$$

Up to relabeling the four branch vertices, the nine affine functional graphs are exactly those in Table 1. All branch arrows have weight one; the critical leaves are supplied by (3.2).

Condition	Branch skeleton	Nonzero indegrees r_v
$4 \mid d$	constant map	4
d even	one fixed root; a length-two tail and one additional vertex entering the middle	$2, 2$
d even	two fixed points, each with one incoming tail	$2, 2$
d even	a two-cycle, with one incoming tail at each cycle vertex	$2, 2$
d odd	four fixed points	$1, 1, 1, 1$
d odd	two fixed points and one two-cycle	$1, 1, 1, 1$
d odd	one fixed point and one three-cycle	$1, 1, 1, 1$
d odd	two two-cycles	$1, 1, 1, 1$
d odd	one four-cycle	$1, 1, 1, 1$

Table 1: Affine branch skeletons on $\mathbb{F}_2^2$. Zero indegrees are omitted in the last column.

To prove exhaustion, classify A by rank. Rank zero gives a constant map. If A has rank one, its image is a two-point affine line, and the restriction to that image is constant, the identity, or the transposition; these are the three nonconstant even-degree rows. If A is invertible, then $\mathrm{AGL}(2, 2)$ is the full symmetric group on four points, whose five cycle types give the five odd-degree rows. Moreover, $d \equiv \det A \pmod 2$, and rank zero modulo two forces $4 \mid d$.

Every row occurs for a rigid map. Given any $A_0 = (a_{ij}) \in M_2(\mathbb{F}_2)$, lift it to

$$M = \begin{pmatrix} a_{11} & a_{12} + 8 \\ a_{21} - 8 & a_{22} \end{pmatrix}. \tag{3.3}$$

Then $M \equiv A_0 \pmod 2$, $\det M > 1$, and

$$(\operatorname{tr} M)^2 - 4 \det M = (a_{11} - a_{22})^2 + 4(a_{12} + 8)(a_{21} - 8) < 0. \tag{3.4}$$

Consequently M is multiplication by a nonreal imaginary-quadratic integer on a suitable invariant lattice. Quotienting by $\{\pm 1\}$ and choosing any $b \in E[2]$ realizes (3.1) by a rigid Lattès map.

4 The signature (3,3,3)

Let $\omega^2 + \omega + 1 = 0$ and $E = \mathbb{C}/\mathbb{Z}[\omega]$. The branch set is

$$K_3 = \ker(1 - \omega) \cong \mathbb{F}_3.$$

Write $\alpha = x + y\omega$ and $c = x + y \pmod 3$. The allowed translation b lies in K_3, and the induced map is

$$\phi(t) = ct + b \qquad (t \in \mathbb{F}_3). \tag{4.1}$$

This yields exactly four skeletons.

All branch arrows have weight one. For $r_v = |\phi^{-1}(v)|$, the target v receives $(d - r_v)/3$ nonbranch critical leaves of weight three. The congruences follow from $d = N(\alpha) \equiv c^2 \pmod 3$. The four rows are realized, for example, by derivatives $1 - \omega$, 4, and 2, with the indicated translations.

Residue data	Branch skeleton	Degree condition
$c = 0$	constant	$3 \mid d$
$c = 1,\ b = 0$	three fixed points	$d \equiv 1 \pmod 3$
$c = 1,\ b \neq 0$	one three-cycle	$d \equiv 1 \pmod 3$
$c = -1$	one fixed point and one two-cycle	$d \equiv 1 \pmod 3$

Table 2: Branch skeletons for signature $(3, 3, 3)$.

5 The signature (2,4,4)

Take $E = \mathbb{C}/\mathbb{Z}[i]$. Let X, Y be the weight-four vertices represented by 0 and $h = (1+i)/2$, and let Z be the weight-two vertex represented by the orbit of $1/2$. The translation is $b \in \{0, h\}$. For $\alpha = x+yi$, direct reduction modulo the Gaussian lattice gives Table 3, up to interchanging X and Y.

Arithmetic case	Branch skeleton	$(\kappa_X, \kappa_Y, \kappa_Z)$
d odd, $b = 0$	X, Y, Z fixed	$((d-1)/4, (d-1)/4, (d-1)/2)$
d odd, $b = h$	$X \leftrightarrow Y$, Z fixed	$((d-1)/4, (d-1)/4, (d-1)/2)$
$d \equiv 0 \pmod 4$	$X, Y, Z \to X$	$((d-4)/4, d/4, d/2)$
$d \equiv 2 \pmod 4$	$Z \to Y \to X \to X$	$((d-2)/4, (d-2)/4, d/2)$

Table 3: The four Gaussian cases.

Every arrow has weight equal to the target weight divided by the source weight. In particular, an arrow $Z \to X$ or $Z \to Y$ has weight two. Multiplication by α sends h to h exactly when x, y have opposite parity, and otherwise sends it to 0. It sends Z to itself when the parities are opposite, to 0 when both are even, and to h when both are odd. Addition of h swaps X, Y and preserves Z. These are precisely the three norm cases in the first column, and Proposition 2.1 gives the displayed leaf counts. The cases occur already with derivatives $1 + 2i$, 2, and $1 + i$ and the allowed translations.

6 The signature (2,3,6)

Again use the Eisenstein torus. Let X, Y, Z have weights $6, 3, 2$, respectively. The vertex X is represented by 0; Y is the orbit of the two nonzero points of $\ker(1 - \omega)$; and Z is the orbit of the three nonzero points of $E[2]$. There is no nonzero allowed translation.

The vertex Y maps to X exactly when $3 \mid d$, and otherwise is fixed. The vertex Z maps to X exactly when $4 \mid d$, and otherwise is fixed. This gives Table 4.

Arithmetic case	Branch skeleton	$(\kappa_X, \kappa_Y, \kappa_Z)$
$3 \nmid d$, $4 \nmid d$	X, Y, Z fixed	$((d-1)/6, (d-1)/3, (d-1)/2)$
$3 \mid d$, $4 \nmid d$	$Y \to X \to X$, Z fixed	$((d-3)/6, d/3, (d-1)/2)$
$3 \nmid d$, $4 \mid d$	$Z \to X \to X$, Y fixed	$((d-4)/6, (d-1)/3, d/2)$
$12 \mid d$	$Y, Z \to X \to X$	$((d-6)/6, d/3, d/2)$

Table 4: The four order-six cases.

Here $Y \to X$ has weight two and $Z \to X$ has weight three. The divisibility tests are reduction of $\alpha \in \mathbb{Z}[\omega]$ modulo the ramified prime above three and modulo two, where $\mathbb{Z}[\omega]/2 \cong \mathbb{F}_4$. The four rows occur, for example, with derivatives $3+\omega$, $1-\omega$, 2, and $2(1-\omega)$, respectively. The leaf counts follow from (2.3).

Sections 3–6, together with Proposition 2.1, exhaust the weighted ramification portraits of all rigid Lattès maps.

7 Every flexible portrait occurs rigidly

Flexible Lattès maps have signature $(2,2,2,2)$, integral derivative $\pm s$, and square degree s^2, where $s \geq 2$. The sign does not alter the action on two-torsion. We now give a rigid map with the same degree and the same complete weighted portrait for every translation.

Fix $s \geq 2$ and put

$$\tau_s = \frac{i}{\sqrt{s-1}}, \qquad E_s = \mathbb{C}/(\mathbb{Z} + \tau_s\mathbb{Z}), \qquad \alpha_s = s - 2 + 2i\sqrt{s-1}. \tag{7.1}$$

Multiplication by α_s preserves the displayed lattice and, in the basis $(1, \tau_s)$, has matrix

$$M_s = \begin{pmatrix} s-2 & -2 \\ 2s-2 & s-2 \end{pmatrix}. \tag{7.2}$$

Consequently

$$\det M_s = s^2, \qquad (\operatorname{tr} M_s)^2 - 4\det M_s = -16(s-1) < 0, \qquad M_s \equiv sI_2 \pmod 2. \tag{7.3}$$

For any $\beta \in E_s[2]$, compare the quotients by $\{\pm 1\}$ induced by

$$z \longmapsto \alpha_s z + \beta \qquad \text{and} \qquad z \longmapsto sz + \beta. \tag{7.4}$$

Both maps have degree s^2, and (7.3) says that they induce the same affine map on $E_s[2]$. Formula (3.2) therefore gives identical complete weighted portraits. The first map is rigid because $\alpha_s \notin \mathbb{Z}$, while the second is flexible.

Theorem 7.1. *Every flexible Lattès ramification portrait, in every possible degree, also occurs for a rigid Lattès map of the same degree.*

More explicitly, for odd s, $\beta = 0$ gives four fixed postcritical points and $\beta \neq 0$ gives two two-cycles; for even s, the postcritical map is constant. The critical-leaf multiplicities agree because the degrees agree.

8 Scope and relation to prior work

The local-degree formula and the Euclidean-orbifold description are standard Lattès theory. Milnor gives the conformal classification and sample critical-orbit diagrams [4]; Parry enumerates small-degree maps in the four-postcritical convention [6]; Floyd–Parry–Pilgrim classify portraits in the much larger topological NET class [3]. A recent degree-three Hessian family supplies a special rigid portrait [5]. We did not locate the four-signature extraction or Theorem 7.1 in those sources.

The classification here is of abstract weighted directed graphs. It does not classify analytic conjugacy classes and does not retain coordinates, cross-ratios, periodic multipliers, or other marked geometric data. Since the result is an elementary extraction from standard torus-quotient theory, the literature status is stated only as apparently unrecorded.

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