

Geometric and Arithmetic Independence of Affine Chebyshev Towers

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Abstract

We determine joint Galois images for finite families of normalized Chebyshev maps with nonconstant affine targets over function fields. Over $\mathbb{C}(t)$, the graph whose edges join the two finite branch points, together with degree parity, determines the joint image when endpoint pairs are distinct. Least common multiples merge repeated pairs, and greatest common divisors and quadratic square classes determine all intersections between subfamilies. This gives exact criteria for strict independence and finite-index independence of the complete preimage towers. Over a number-field function field $k(t)$, pairwise coprime finite degrees give an arithmetic image obtained by multiplying the graph cut-space signs by the cyclotomic Galois image. We determine the full constant field and show that the joint index divides $[k : \mathbb{Q}]$. For complete towers, finite arithmetic index is equivalent to global pairwise coprimality of the degrees. Over $\mathbb{Q}(t)$ this is equivalent to strict independence, even when the geometric fields are entangled. These results concern the stated Chebyshev family, not arbitrary rational maps or constant targets over number fields. This manuscript is unrefereed.

1 The independence question and the two classifications

Ingram's independence question in the AIM Galois theory and arithmetic dynamics problem list [1, Problem 15.1] asks when the Galois actions on preimages of two points are independent. This is a broad question. We treat nonconstant affine targets for Chebyshev maps over function fields, allowing several maps and several targets. Even within this family, geometric independence and arithmetic independence have different criteria.

Normalize the Chebyshev polynomials by

$$C_0 = 2, \quad C_1 = x, \quad C_{m+1} = xC_m - C_{m-1}.$$

They satisfy

$$C_m(u + u^{-1}) = u^m + u^{-m}, \quad C_a \circ C_b = C_{ab}. \quad (1)$$

For a field $k \subseteq \mathbb{C}$, let $K = k(t)$ and fix a finite nonempty list of targets

$$\ell_i(t) = A_i t + B_i, \quad A_i \in k^\times, \quad B_i \in k. \quad (2)$$

Let $L_i(m)/K$ be the splitting field of $C_m(x) - \ell_i(t)$. All fields are taken in a common algebraic closure. For degrees $d_i \geq 2$, set

$$L_{i,\infty} = \bigcup_{n \geq 1} L_i(d_i^n).$$

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These are nested fields by (1). Their joint image is the image of restriction from the Galois group of their compositum to the product of their individual Galois groups. We call the family *strictly independent* if this image is the whole product. Finite-index independence means that its index is finite. For finite Galois extensions, strict independence is joint linear disjointness.

The endpoints attached to target i are

$$a_i = \frac{-2 - B_i}{A_i}, \quad b_i = \frac{2 - B_i}{A_i}. \quad (3)$$

They are distinct. The resulting endpoint graph has one edge for each target and no isolated vertices; parallel edges are allowed. The group D_m denotes the dihedral group of order $2m$.

Theorem 1.1 (Geometric towers). *Over $\mathbb{C}(t)$, group the target indices into classes P with the same unordered endpoint pair. Form the simple graph of these classes. Let r be the number of original targets, v its number of vertices, $D_P = \text{lcm}_{i \in P} d_i$, g_2 the number of even D_P , and c_o the number of connected components all of whose D_P are odd. At every common level $n \geq 2$, the joint index is*

$$I_n = 2^{r+g_2-v+c_o} \left(\frac{\prod_i d_i}{\prod_P D_P} \right)^n. \quad (4)$$

The infinite joint image has finite index if and only if the degrees within each endpoint class are pairwise coprime. In that case the index is $2^{r+g_2-v+c_o}$. Strict independence holds if and only if there are no repeated endpoint pairs, the graph is a forest, and each connected component contains at most one even-degree edge.

Theorem 1.2 (Arithmetic towers). *Let k be a number field. The towers in (2) over $k(t)$ have finite-index joint arithmetic image if and only if the degrees d_i are globally pairwise coprime. When finite, the index divides $[k : \mathbb{Q}]$. Over $\mathbb{Q}(t)$, finite-index independence is therefore equivalent to strict independence. These assertions allow arbitrary overlaps of the endpoint pairs.*

The difference between the two criteria is essential. A shared prime in two degrees produces common real cyclotomic constants of unbounded degree over a number field. Conversely, adding the arithmetic cyclotomic action can remove a finite geometric dependence when the degrees are coprime. We give the precise finite arithmetic group in Theorem 4.1.

The ingredients are classical. Dihedral monodromy and real cyclotomic constants of prime-degree Dickson covers appear in Guralnick–Müller–Saxl [2, Section 5, Proposition 5.4]. Klahn–König [3, Section 5.3] use arithmetic Chebyshev groups and quadratic subextensions to construct joint images in a different problem about local value sets. We do not claim those individual-cover calculations or the quadratic-entanglement method as new. The purpose here is the explicit endpoint-graph classification, its field intersections, and the arithmetic cut-space formula for the stated families. No absolute priority claim is made.

2 Individual covers and the geometric joint image

Throughout this section $K = \mathbb{C}(t)$ and each finite degree m_i is at least three. Choose u_i satisfying

$$u_i^{m_i} + u_i^{-m_i} = \ell_i(t). \quad (5)$$

The extension $\mathbb{C}(u_i)/K$ has degree $2m_i$. The transformations $u_i \mapsto \zeta_i u_i$ and $u_i \mapsto u_i^{-1}$, for a primitive m_i -th root of unity ζ_i , give its dihedral Galois group. The polynomial roots are

$$x_{i,j} = \zeta_i^j u_i + \zeta_i^{-j} u_i^{-1}, \quad 0 \leq j < m_i. \quad (6)$$

The two equations for $j = 0, 1$ recover u_i, u_i^{-1} because their determinant is $\zeta_i^{-1} - \zeta_i \neq 0$. Consequently $L_i(m_i) = \mathbb{C}(u_i)$.

The only finite branch points are a_i, b_i , with reflection inertia. The inertia at infinity has order m_i . The two finite branch loops generate the dihedral group. In particular their product is a primitive rotation. Write

$$D_{m_i} = \langle r_i, s_i \mid r_i^{m_i} = s_i^2 = 1, s_i r_i s_i = r_i^{-1} \rangle.$$

Its derived subgroup is $N_i = \langle r_i^2 \rangle$, of order $\delta(m_i) = m_i / \gcd(m_i, 2)$. Its maximal abelian subfield is

$$F_i = \begin{cases} K(\sqrt{(t-a_i)(t-b_i)}), & m_i \text{ odd}, \\ K(\sqrt{t-a_i}, \sqrt{t-b_i}), & m_i \text{ even}. \end{cases} \quad (7)$$

Indeed, in the odd case $u_i^{m_i} - u_i^{-m_i}$ has square $\ell_i(t)^2 - 4$ and is fixed by rotations. In the even case the squares of $u_i^{m_i/2} + u_i^{-m_i/2}$ and $u_i^{m_i/2} - u_i^{-m_i/2}$ are $\ell_i + 2$ and $\ell_i - 2$. Constants are squares in $\mathbb{C}$, and endpoint valuations give the claimed quadratic ranks. The degrees equal the abelianization orders.

Lemma 2.1 (Derived subgroup isolation). *If the endpoint pairs are distinct and $L = \prod_i L_i(m_i)$, then*

$$N := \prod_i N_i \subseteq H := \text{Gal}(L/K), \quad [H, H] = N.$$

The joint group H is the full inverse image of the restriction image of $\text{Gal}(F/K)$ in the product of the individual abelianizations, where $F = \prod_i F_i$. Moreover F is the maximal abelian subfield of L/K .

Proof. Choose a geometric system of loops about all vertices in the punctured complex affine line, and denote their joint monodromy elements by g_x . In coordinate i , g_x is the identity unless x is an endpoint of edge i , in which case it is a reflection. Filling the other punctures shows that the two endpoint reflections still generate D_{m_i} . The inverse product of the finite loops accounts for the loop at infinity.

For edge $i = \{a, b\}$, the commutator $[g_a, g_b]$ is supported only on coordinate i : every other edge misses at least one endpoint. There it is the square or inverse square of a primitive rotation. It therefore generates N_i . This proves $N \subseteq [H, H]$. Since the quotient of H by N embeds in a product of abelianizations, equality follows. Restriction to F has kernel N and is onto its stated image; automorphisms of F extend to the normal field L . The full-inverse-image assertion and the maximal abelian assertion follow. $\square$

Let $G = (V, E)$ be the simple endpoint graph, with $e = |E|$ and $v = |V|$. Let e_2 be its number of even-degree edges, and let c_o count its all-odd connected components. In $K^\times / K^{\times 2}$ the classes $[t - x]$, $x \in V$, are independent, as their finite valuations show. Identify their span with $\mathbb{F}_2^V$, with basis ϵ_x . Let W be generated by

$$\epsilon_{a_i} + \epsilon_{b_i} \quad (m_i \text{ odd}), \quad \epsilon_{a_i}, \epsilon_{b_i} \quad (m_i \text{ even}). \quad (8)$$

This is the square-class space of F .

Theorem 2.2 (Distinct endpoint pairs). *For distinct endpoint pairs,*

$$\left[\prod_i D_{m_i} : \text{Gal}(L/K) \right] = 2^{e+e_2-v+c_o}, \quad [L : K] = \left(\prod_i \delta(m_i) \right) 2^{v-c_o}. \quad (9)$$

The family is strictly independent if and only if G is a forest with at most one even-degree edge in each connected component.

Proof. A vector orthogonal to W has equal endpoint coordinates on each odd edge and zero endpoint coordinates on each even edge. It is constant on an all-odd component and vanishes on a component containing an even edge. Thus $\dim W = v - c_o$. Lemma 2.1 gives the degree formula, and division by the product of individual degrees gives the index. In a connected all-odd component the exponent is $e - v + 1$. In a component containing $h \geq 1$ even edges it is $(e - v + 1) + (h - 1)$. Both summands are nonnegative; vanishing is exactly the stated forest condition. $\square$

Corollary 2.3 (Disjoint subfamilies). *Partition the edge set into $A \sqcup B$, with empty composita interpreted as K . Let W_A, W_B be the corresponding spaces in $\mathbb{F}_2^V$, and define L_A, F_A, L_B, F_B accordingly. Then*

$$L_A \cap L_B = F_A \cap F_B = K(\sqrt{h} : [h] \in W_A \cap W_B). \quad (10)$$

Proof. The geometric degree formula applies to both subfamilies. Since $W_{A \cup B} = W_A + W_B$, cancellation in $[L_A \cap L_B : K] = [L_A : K][L_B : K]/[L : K]$ gives $2^{\dim(W_A \cap W_B)}$. The multiquadratic field on the right is contained in the intersection and has this degree, so equality holds. $\square$

3 Repeated endpoint pairs and exact intersections

For one endpoint pair and a choice of its affine function ℓ , use the notation

$$E_q = K(u_q), \quad u_q^q + u_q^{-q} = \ell(t), \quad q \geq 1.$$

This denotes the degree- $2q$ radical cover even when $q = 1, 2$. For these two degrees it is *not* the splitting field of $C_q(x) - \ell(t)$. Its group is still D_q , interpreting $D_1 = C_2$ and $D_2 = C_2^2$. Its maximal abelian field follows the same odd/even rule as (7). The argument in Lemma 2.1 applies to distinct pairs of these auxiliary covers: when $q = 1, 2$ the derived subgroup is trivial.

Two affine functions have the same unordered endpoint pair exactly when they differ by a sign. Changing ℓ to $-\ell$ does not change E_q over $\mathbb{C}(t)$: replace u_q by ηu_q with $\eta^q = -1$. If $Q = \text{lcm}_j q_j$, choose $u_{q_j} = u_Q^{Q/q_j}$. Since $\gcd_j(Q/q_j) = 1$, Bezout's identity recovers u_Q from these powers. Therefore

$$\prod_j E_{q_j} = E_{\text{lcm}_j q_j}, \quad E_a \cap E_b = E_{\gcd(a,b)}. \quad (11)$$

For the second equality, the field on the right is contained in the intersection, and the compositum formula gives the same degree $2 \gcd(a, b)$. Normality makes these identities independent of the initial choices of conjugate radicals.

For an arbitrary list of r targets of degrees $m_i \geq 3$, group equal endpoint pairs into classes P and set $M_P = \text{lcm}_{i \in P} m_i$. Write g for the number of classes, g_2 for the number of even M_P , v for the number of endpoints, and c_o for the number of all-odd components in the merged simple graph. Applying Theorem 2.2 after (11) gives

$$\left[\prod_{i=1}^r D_{m_i} : \text{Gal} \left(\prod_i L_i(m_i) / K \right) \right] = 2^{r+g_2-v+c_o} \frac{\prod_i m_i}{\prod_P M_P}. \quad (12)$$

In particular, a repeated pair contributes an index of at least two.

There is also an exact common-field formula. Partition the original index list into $A \sqcup B$. For a class appearing in both parts put

$$M_{A,P} = \text{lcm}_{i \in A \cap P} m_i, \quad M_{B,P} = \text{lcm}_{i \in B \cap P} m_i, \quad q_P = \gcd(M_{A,P}, M_{B,P}).$$

Form W_A, W_B by (8) on the respective merged graphs, and set

$$F_I = K(\sqrt{h} : [h] \in W_A \cap W_B), \quad R_I = \prod_{P \text{ common}} E_{q_P, P}. \quad (13)$$

The auxiliary convention for $q_P = 1, 2$ is essential here.

Theorem 3.1 (All geometric intersections). *With the preceding notation,*

$$L_A \cap L_B = R_I F_I, \quad [L_A \cap L_B : K] = \left(\prod_{P \text{ common}} \delta(q_P) \right) 2^{\dim(W_A \cap W_B)}. \quad (14)$$

Proof. Equation (11) shows that $R_I \subseteq L_A \cap L_B$; the inclusion of F_I is immediate. The identity

$$\delta(a)\delta(b) = \delta(\gcd(a, b))\delta(\text{lcm}(a, b))$$

follows prime by prime: at the prime two, replace a valuation s by $\max(s - 1, 0)$; at other primes leave it unchanged. Applying the degree formula to the three merged graphs for $A, B, A \cup B$ therefore gives the degree on the right of (14). The square-class space of the union is $W_A + W_B$, including for a shared pair, since the least common multiple is even precisely when at least one input degree is even.

Let F_R be the maximal abelian subfield of R_I . The auxiliary version of Lemma 2.1 gives $[R_I : F_R] = \prod_{P \text{ common}} \delta(q_P)$. Since $R_I \subseteq L_A \cap L_B$, the abelian extension F_R/K lies in both maximal abelian subfields F_A, F_B , and thus in F_I . Conversely $R_I \cap F_I$ is abelian over K and lies in F_R . Hence $R_I \cap F_I = F_R$, and $R_I F_I$ has exactly the required degree. The inclusion already proved is consequently equality. $\square$

Proof of Theorem 1.1. At common depth $n \geq 2$, every polynomial degree is at least three and $M_P = D_P^n$. Formula (12) becomes (4); the parities and the graph remain fixed. Prime by prime, $\prod_{i \in P} d_i / \text{lcm}_{i \in P} d_i$ is an integer, equal to one precisely when those degrees are pairwise coprime. Thus the finite indices are constant in the asserted case and tend to infinity otherwise.

For a closed subgroup of a profinite group, its index is the supremum of its indices in a cofinal family of finite quotients. Indeed, any finite collection of distinct cosets can be separated in a sufficiently fine finite quotient, using closedness and compactness. Equal-depth levels are cofinal here. This proves the finite-index criterion. A repeated pair precludes index one by (12); without repetitions, Theorem 2.2 gives the strict criterion. $\square$

For completeness, Theorem 3.1 also determines the intersection of infinite subfamilies. Put

$$D_{A,P} = \text{lcm}_{i \in A \cap P} d_i, \quad D_{B,P} = \text{lcm}_{i \in B \cap P} d_i, \quad c_P = \gcd(D_{A,P}, D_{B,P}).$$

The space $W_A \cap W_B$ is unchanged at all depths $n \geq 2$, and

$$L_{A,\infty} \cap L_{B,\infty} = F_I \prod_{P \text{ common}} \bigcup_{n \geq 2} E_{c_P^n, P}. \quad (15)$$

Every element in the intersection lies at a finite common depth, so this follows by taking unions in (14). If $c_P = 1$, its factor is the fixed quadratic field $E_{1,P}$, already in F_I . If some $c_P \geq 2$, the common field contains extensions of degrees $2c_P^n$ for all n . Thus the intersection is finite exactly when all these c_P equal one. For distinct endpoint pairs it is just the finite multiquadratic field F_I .

4 The finite arithmetic image

Now let $k \subset \mathbb{C}$ be a number field, $K = k(t)$, and suppose the finite degrees $m_i \geq 3$ are globally pairwise coprime. No restriction is placed on the endpoint overlaps. In the endpoint multigraph let v be the number of vertices, c the number of components, and $b = v - c$. Its cut space is

$$\mathcal{C} = \{(z_{a_i} + z_{b_i})_{i=1}^r : z \in \mathbb{F}_2^V\} \subseteq \mathbb{F}_2^r, \quad (16)$$

of dimension b . Define

$$\begin{aligned} M &= \prod_i m_i, & \Omega &= k(\zeta_M), & U &= \prod_i (\mathbb{Z}/m_i\mathbb{Z})^\times, \\ \Gamma &= \text{Gal}(\Omega/k) \hookrightarrow U, & S &= \{((-1)^{s_i})_i : s \in \mathcal{C}\}, & J &= \Gamma \cap S, \\ B &= \Gamma S, & R &= \prod_i \mathbb{Z}/m_i\mathbb{Z}. \end{aligned} \quad (17)$$

The map from cut vectors to signs is injective since $m_i \geq 3$. Let Γ_i be the coordinate image of Γ and put $B_i = \Gamma_i\{1, -1\}$. Here $L_i = L_i(m_i)$ and $L = \prod_i L_i$.

Theorem 4.1 (Arithmetic cut-space formula). *In the root labeling (6),*

$$\text{Gal}(L/K) = R \rtimes B, \quad (q, z) : j \mapsto q_j + z_j j \quad \text{in coordinate } i. \quad (18)$$

The algebraic closure of k in L is Ω^J . If $F_i^{\text{const}} = k(\zeta_{m_i} + \zeta_{m_i}^{-1})$, then

$$[L : K] = M \frac{2^b [\Omega : k]}{|J|}, \quad (19)$$

$$I := \left[\prod_i \text{Gal}(L_i/K) : \text{Gal}(L/K) \right] = \frac{\prod_i |B_i|}{|B|} = 2^{r-b} \frac{|J| \prod_i [F_i^{\text{const}} : k]}{[\Omega : k]}. \quad (20)$$

In particular,

$$I \mid [k \cap \mathbb{Q}(\zeta_M) : \mathbb{Q}] \mid [k : \mathbb{Q}]. \quad (21)$$

Over $\mathbb{Q}(t)$ the fields L_i are jointly linearly disjoint.

The full constant field Ω^J need not be the compositum of the individual constant fields. The proof below keeps that distinction.

4.1 Regularity after adjoining roots of unity

Choose u_i as in (5) and $\zeta_i = \zeta_M^{M/m_i}$. Set $E_i = \Omega(u_i)$ and $E = \prod_i E_i$. Each $E_i/\Omega(t)$ is a regular rational dihedral cover. Recovering u_i from two polynomial roots shows $E = L\Omega$, so E/K is Galois.

The geometric joint group contains $N = \prod_i D'_{m_i}$ even with parallel edges. The commutator about an endpoint pair is supported precisely on the parallel class. Its coordinate orders are $m_i/\text{gcd}(m_i, 2)$, which are pairwise coprime. Powers of that commutator isolate every coordinate generator by the Chinese remainder theorem. This is the modification of Lemma 2.1 needed for the multigraph.

For odd m_i the abelian subfield of $E_i/\Omega(t)$ is generated by

$$w_i = u_i^{m_i} - u_i^{-m_i}, \quad w_i^2 = A_i^2(t - a_i)(t - b_i).$$

There is at most one even degree, say at index e . Its two radicals are

$$\begin{aligned} f_e &= u_e^{m_e/2} + u_e^{-m_e/2}, & f_e^2 &= A_e(t - a_e), \\ g_e &= u_e^{m_e/2} - u_e^{-m_e/2}, & g_e^2 &= A_e(t - b_e), \end{aligned}$$

and $f_e g_e = w_e$. Let F be the compositum of these abelian fields, including Ω .

A product of the displayed radicands is a square over $\mathbb{C}(t)$ only if every endpoint valuation is even. The odd-edge radicands have degree two; the even-edge radicands have degree one. Thus such a relation selects both even radicals or neither. Replacing their joint selection by the even edge, its selected edges form a cycle-space vector. Every selected edge contributes the slope factor A_i^2 . The entire product is consequently already a square in $k(t)$, not just over algebraically closed constants.

The quadratic rank is therefore unchanged after extending constants from Ω to $\mathbb{C}$. It is b with no even edge and $b + 1$ with one even edge. Hence F is regular over Ω . The group of $E/\Omega(t)$ and its geometric subgroup both contain N , and their quotients by N are the respective images on F . These have equal order. The groups coincide, proving that E is regular over Ω and

$$[E : \Omega(t)] = M2^b. \quad (22)$$

Use geometric coordinates (q, s) in which root indices transform by $j \mapsto q_i + (-1)^{s_i} j$. The sign on an odd radical is $(-1)^{s_i}$. The signs on f_e, g_e are $(-1)^{q_e}$ and $(-1)^{q_e + s_e}$. The cycle relations constrain exactly $s \in \mathcal{C}$ and do not constrain translation coordinates, including the possible even parity. Thus

$$\text{Gal}(E/\Omega(t)) = R \rtimes \mathcal{C}. \quad (23)$$

4.2 Compatible semilinear lifts

Lemma 4.2. *Every $\tau \in \Gamma$ has a unique lift $h_\tau \in \text{Gal}(E/K)$ that fixes every u_i . These lifts form a group section.*

Proof. On each rational field $E_i = \Omega(u_i)$, acting by τ on Ω and fixing u_i defines a K -automorphism. It fixes the quadratic radicals displayed above. These prescribed maps agree on F : every multiplicative relation between the radical generators has value $\pm h(t)$ for some $h(t) \in k(t)$, by the preceding square-class calculation. Choose a square-class basis; it therefore defines a semilinear automorphism of F fixing all these generators.

Extend this K -automorphism of F to $\beta \in \text{Gal}(E/K)$. Each E_i is normal over K and is preserved by β . In coordinate i , the desired map and $\beta|_{E_i}$ agree on Ω and on its maximal abelian subfield. Their quotient belongs to N_i . Since the whole product N belongs to the geometric joint group, one can simultaneously correct β by these quotients. The result is the required lift. Uniqueness follows because E is generated by Ω and all u_i . Compositions have the same prescribed action, so the lifts form a homomorphic section. $\square$

Conjugation by h_τ multiplies translation coordinate i by the cyclotomic exponent α_i of τ and leaves the reflection bits unchanged. Hence

$$\text{Gal}(E/K) = R \rtimes (\mathcal{C} \times \Gamma). \quad (24)$$

The root action of (q, s, α) is

$$j \mapsto q_i + (-1)^{s_i} \alpha_i j \pmod{m_i}. \quad (25)$$

For example, one consistent convention on radicals is $\sigma(u_i) = \zeta_i^{\varepsilon_i q_i} u_i^{\varepsilon_i}$, $\sigma(\zeta_i) = \zeta_i^{\alpha_i}$, where $\varepsilon_i = (-1)^{s_i}$. Substitution in (6) gives (25).

4.3 The root kernel and constant field

Proof of Theorem 4.1. An element in (25) fixes all roots precisely when $q_i = 0$ and $\alpha_i = (-1)^{s_i}$ for every i , as testing indices zero and one shows. Its kernel is

$$\{(0, s, \text{sgn}(s)) : s \in \mathcal{C}, \text{sgn}(s) \in \Gamma\} \cong J.$$

This kernel is central: its multiplier on R is trivial, and $\mathcal{C} \times \Gamma$ is abelian. Quotienting (24) gives $R \rtimes \Gamma S$ and its asserted root action.

Because E is regular over Ω , the constants of L are $L \cap \Omega$. The kernel just computed has restriction image J on Ω , so this intersection is Ω^J . This also gives (19).

For a single edge the cut space supplies both signs. Its root group is $(\mathbb{Z}/m_i\mathbb{Z}) \rtimes B_i$. Within $k(\zeta_{m_i})$, the subgroup fixing $\zeta_{m_i} + \zeta_{m_i}^{-1}$ is precisely $\Gamma_i \cap \{1, -1\}$. Thus the individual constant field is F_i^{const} and $|B_i| = 2[F_i^{\text{const}} : k]$. Taking the quotient of the individual degree product by (19) proves (20).

Finally, the subgroup chain

$$\Gamma \leq \Gamma S \leq \prod_i B_i \leq U$$

shows that I divides $[U : \Gamma]$. Chinese remaindering identifies U with $\text{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$, and restriction gives $[U : \Gamma] = [k \cap \mathbb{Q}(\zeta_M) : \mathbb{Q}]$. This proves (21). When $k = \mathbb{Q}$, $\Gamma = U$ and the joint root image is the entire product. $\square$

5 Arithmetic towers and examples

Proof of Theorem 1.2. For pairwise coprime degrees, apply (21) at the cofinal levels $m_i = d_i^n$, $n \geq 2$. Their indices divide the fixed integer $[k : \mathbb{Q}]$. They increase by divisibility, since the product and joint Galois groups both surject to lower levels. They therefore stabilize. The profinite index argument used in the geometric proof gives the same divisor bound for the infinite image.

Conversely, if p divides d_i and d_j , their level- n constant fields both contain

$$k(\zeta_{p^{e_n}} + \zeta_{p^{e_n}}^{-1}), \quad e_n = n \min(v_p(d_i), v_p(d_j)).$$

For all sufficiently large n , this has degree at least $\varphi(p^{e_n})/(2[k : \mathbb{Q}])$ over k , which tends to infinity. The individual constant-field calculation applies without any coprimality assumption on other coordinates, since it is the one-edge case of Theorem 4.1. For two finite Galois extensions, the index of the joint image is the degree of their intersection. The pairwise indices are thus unbounded, and the full joint index is at least each pairwise index. This proves necessity. The assertion over $\mathbb{Q}$ follows from the divisor bound. $\square$

In particular, two affine targets for the same map C_d , $d \geq 2$, always have infinite-index joint arithmetic image over $k(t)$ for a number field k . This remains true when their geometric joint image has finite index.

Example 5.1 (Pairwise does not imply joint geometric independence). Over $\mathbb{C}(t)$ use the targets $4t - 2$, $4t - 6$, and $2t - 2$. Their endpoint graph is the triangle on $0, 1, 2$. For any fixed odd degree $d \geq 3$, the three C_d towers are pairwise independent but have joint index two. Each two-edge subgraph is a tree; the triangle has one cycle. At depth n the joint degree is $4d^{3n}$, instead of $8d^{3n}$. The quadratic relation is the product of $t(t - 1)$, $(t - 1)(t - 2)$ and $t(t - 2)$ being a square.

Example 5.2 (A tree obstruction from parity). On the path $0 - 1 - 2 - 3$ use targets $4t - 2$, $4t - 6$, $4t - 10$ with maps C_2, C_3, C_2 . Every pair of geometric towers is independent. The whole tree contains two even edges, so its joint index is two. The relation couples the middle odd edge with the endpoint square classes supplied by the two outer even edges.

Example 5.3 (Arithmetic independence with coincident endpoints). Over $\mathbb{Q}(t)$, degrees three and five at the same affine target are strictly independent arithmetically. At the finite levels $m_1 = 3, m_2 = 5$, the joint degree is 120, the geometric degree is 30, and the constant field is the real subfield of $\mathbb{Q}(\zeta_{15})$, of degree four. The geometric towers share the endpoint quadratic field, but the arithmetic towers are strictly independent by Theorem 1.2.

Example 5.4 (Additional constants around a triangle). Assign degrees three, five and seven to the triangle targets above, now over $\mathbb{Q}(t)$. The finite arithmetic degree is 5040, the geometric degree is 420, and the constant field has degree twelve. The compositum of the individual real cyclotomic constant fields has degree six. Thus the full compositum acquires an additional constant extension. Its arithmetic root image is nevertheless the full product.

Example 5.5 (A nontrivial number-field index). Let $k = \mathbb{Q}(\sqrt{10})$, and take maps C_5, C_2 with any two nonconstant affine targets over k . At common levels $n \geq 3$ their individual constants contain $k(\sqrt{5})$ and $k(\sqrt{2})$, which are the same quadratic extension of k . The joint index is therefore at least two. Theorem 1.2 bounds it by two, so the infinite joint index is exactly two.

6 Scope and reproducibility

The geometric classification treats all degree lists and nonconstant affine targets over $\mathbb{C}(t)$. The finite arithmetic formula assumes pairwise coprime degrees; the arithmetic tower criterion does not. Finite polynomial statements require degrees at least three, with cofinal levels covering degree-two towers. There is no specialization theorem for constant number-field targets and no classification for arbitrary rational maps or targets. The general AIM question remains open.

The standard-library Python checks cover graph ranks, repeated-pair intersections and arithmetic groups: 22,061 geometric conditions on 841 configurations, and 646,484 arithmetic conditions including 16 explicit groups and six negative controls. Normal and optimized runs agree. These finite checks do not prove field regularity, semilinear gluing or inverse-limit assertions; those require the written arguments above.

AI assistance was used for mathematical exploration, source comparison, exact computation and exposition. The arguments received project-level self-audit, not independent human peer review or formal proof-assistant verification. Corrections and prior-work pointers are welcome.

References

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