Maximal Finite-Set Stabilizers in Thompson's Group T

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Abstract

Let $\mathbb{D} = \mathbb{Z}[1/2]/\mathbb{Z}$ be the dyadic circle and let T be Thompson's orientation-preserving circle group. We prove that, for every nonempty finite set $A \subset \mathbb{D}$, its setwise stabilizer is a maximal proper subgroup of countably infinite index in T. If $|A| = k$, then

$$\mathrm{Stab}_T(A) \cong F^k \rtimes C_k = F \wr C_k,$$

where the cyclic group permutes the factors. Stabilizers with the same cardinality are conjugate, whereas those with distinct cardinalities are pairwise nonisomorphic. The proof establishes a general circular-order criterion: a group with the finite circular extension property acts primitively on the k-element subsets for every k. This yields a countably infinite family of maximal infinite-index subgroups and supplies examples requested in the 2024 AIM problem list on groups of dynamical origin. The theorem is apparently unrecorded in the literature checked; no absolute priority claim is made.

1 Introduction

The 2024 AIM workshop *Groups of dynamical origin* asked for new maximal subgroups of infinite index in Thompson groups [1, 2]. For Thompson's interval group F, point stabilizers and finite-set stabilizers have been studied extensively [7, 9]. For V, finite point-set stabilizers arise from high transitivity and type systems [4]; finite-set stabilizers have also been characterized in broad classes of ample groups acting on zero-dimensional spaces [8]. For the circle group T, the point stabilizer $F < T$ is familiar, and T itself is maximal in V [3]. We consider instead all finite subsets of the dyadic orbit in the circle action of T.

The obstacle to importing the argument for V is that T preserves circular order and is not highly transitive in the usual ordered sense. Nevertheless, every finite cyclic-order-preserving partial map of the dyadic circle extends to T. We show that this weaker homogeneity still forces primitivity on unordered k-sets. The key is a connected graph of one-point moves. Every edge of that graph can be replaced by a path of length two in every non-diagonal orbital graph.

Our main result is the following.

Theorem 1.1. *Let $A \subset \mathbb{D}$ be finite and nonempty, with $|A| = k$. Then $H_A = \mathrm{Stab}_T(A)$ is maximal proper in T and*

$$[T : H_A] = \aleph_0, \qquad H_A \cong F^k \rtimes C_k = F \wr C_k.$$

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The standard piecewise-linear and tree-pair models of F and T are reviewed in [6]. We use only elementary consequences of those models. Section 2 proves the general circular-order criterion, Section 3 applies it to T, and Section 4 identifies the stabilizers.

2 A circular-order primitivity criterion

Let X be an infinite dense circular order. Say that $G \le \mathrm{Aut}(X)$ has the *finite circular extension property* if every cyclic-order-preserving bijection between finite subsets of X extends to an element of G. Write $\Omega_k = \binom{X}{k}$.

For $P, Q \in \Omega_k$, call P, Q a *local-move pair* if $|P \cap Q| = k - 1$ and the two exceptional points lie in the same circular gap of $P \cap Q$. When $k = 1$, every pair is declared local. Let L_k be the graph of all local moves.

Lemma 2.1. *The graph L_k is connected. Any two vertices are joined by a path of length at most $2k$.*

Proof. Choose a cut outside two given vertices P, Q and list their points in the resulting linear order:

$$p_1 < \cdots < p_k, \qquad q_1 < \cdots < q_k.$$

Set $m_i = \min(p_i, q_i)$. Since both lists are strictly increasing, so is (m_i). Starting at P, replace p_i by m_i, whenever they differ, in the order $i = 1, \dots, k$. At the moment of replacement, both points lie strictly between the same two neighboring points of the other $k-1$ points, so the step is local. This reaches $M = \{m_1, \dots, m_k\}$ in at most k steps. From M, replace m_i by q_i in reverse order $i = k, \dots, 1$. The same inequalities show that every step is local, and at most k further steps reach Q. □

For a transitive action on a set Ω, choose a non-diagonal orbit $\Delta \subset \Omega \times \Omega$. Its *orbital graph* has vertex set Ω and undirected edge set $\Delta \cup \Delta^*$. The action is primitive if and only if every non-diagonal orbital graph is connected: a nontrivial block disconnects the orbital defined by two of its points, while the connected components of a disconnected orbital graph form a nontrivial invariant partition.

Lemma 2.2. *Suppose G has the finite circular extension property. Every edge of L_k has distance at most two in every non-diagonal orbital graph of G on Ω_k.*

Proof. For $k = 1$, finite extension makes the unique non-diagonal orbital graph complete. Assume $k \ge 2$. Fix an ordered-pair orbit represented by (U, V) with $U \ne V$, and let P, Q be a local-move pair. Write

$$S = P \cap Q, \qquad P = S \cup \{x\}, \qquad Q = S \cup \{y\}.$$

Choose $u \in U \setminus V$. A cyclic-order isomorphism $U \to P$ may be chosen with $u \mapsto x$. Replace this one image by $u \mapsto y$ and keep all other images in S; because x and y occupy the same gap of S, this also gives a cyclic-order isomorphism $U \to Q$. In both maps, the images of $U \cap V$ belong to S.

We now construct one k-set C that completes both partial maps to the cyclic type of (U, V). That type records, in each gap of U, the ordered points of $V \setminus U$. All gaps not adjacent to u are identical for P and Q, so place the prescribed points there. Let (a, b) be the gap of S containing

x, y, written in a lifted linear order. If $a < x < y < b$, put the points prescribed immediately before u in (a, x) and those prescribed immediately after u in (y, b). If $a < y < x < b$, use (a, y) and (x, b). Density provides any required finite number of distinct points. Add the already determined images of $U \cap V$.

Thus (P, C) and (Q, C) have the same circular order and equality pattern as (U, V). The extension property puts both ordered pairs in Δ. Therefore $P - C - Q$ is a path of length two in the chosen orbital graph. $\square$

Theorem 2.3 (Circular extension criterion)**.** *If G has the finite circular extension property, then its action on $\binom{X}{k}$ is primitive for every $k \geq 1$. Every non-diagonal orbital graph has diameter at most $4k$.*

Proof. Lemma 2.1 joins any two vertices by at most $2k$ local moves. Lemma 2.2 replaces each local move by at most two edges in any chosen non-diagonal orbital graph. Hence every such graph is connected, with the stated diameter bound, and the action is primitive. $\square$

3 The dyadic circle action of T

Put $S^1 = \mathbb{R}/\mathbb{Z}$ and $\mathbb{D} = \mathbb{Z}[1/2]/\mathbb{Z}$. In the standard model, T consists of the orientation-preserving piecewise-linear homeomorphisms of S^1 with finitely many dyadic breakpoints, dyadic images of breakpoints, and slopes integral powers of two [6].

Lemma 3.1. *Every cyclic-order-preserving bijection between finite subsets of $\mathbb{D}$ extends to an element of T.*

Proof. List the source and target sets in corresponding cyclic order. Each arc between consecutive marked points has dyadic endpoints and is therefore a finite union of standard dyadic intervals. Refine the source and target partitions until they contain the same number of pieces; a bisection raises the number of pieces by one. Map corresponding pieces affinely and in order. Each slope is a quotient of powers of two and hence a power of two, and every breakpoint and image is dyadic. Performing this construction in each complementary arc gives an element of T extending the bijection. $\square$

Lemma 3.1 and Theorem 2.3 show that T acts primitively on $\binom{\mathbb{D}}{k}$. In a transitive action, subgroups between a point stabilizer and the whole group correspond to blocks containing that point. Hence the setwise stabilizer H_A of any $A \in \binom{\mathbb{D}}{k}$ is maximal proper in T.

The orbit–coset bijection

$$T/H_A \longrightarrow \binom{\mathbb{D}}{k}, \qquad gH_A \longmapsto gA,$$

shows that $[T : H_A] = \aleph_0$. The same extension lemma maps any dyadic k-set to any other, so all H_A with fixed k are conjugate.

4 The abstract stabilizer

Order the points $a_0, \ldots, a_{k-1}$ of A cyclically, and let I_i be the closed arc from a_i to a_{i+1}, with indices modulo k. The kernel K_A of the action of H_A on A fixes every endpoint and restricts independently to the k arcs. A dyadic piecewise-linear identification of any I_i with $[0,1]$ conjugates its endpoint-fixing Thompson-like group to F; this standard "F-in-a-box" fact is also recorded in [5]. The restrictions glue at their fixed endpoints, so

$$K_A \cong F^k. \tag{4.1}$$

The induced permutation of A preserves cyclic order and is therefore cyclic. Every cyclic rotation is realized by Lemma 3.1. Consequently,

$$1 \longrightarrow F^k \longrightarrow H_A \longrightarrow C_k \longrightarrow 1 \tag{4.2}$$

is exact. It splits. Indeed, choose a full finite binary tree with k leaves. The tree-pair with the same tree in domain and range and with the leaf intervals cyclically shifted defines an element of T of order k. Its interval-boundary set has size k. Conjugating by Lemma 3.1 makes that boundary set equal to A. Choosing the identifications of successive arcs equivariantly, conjugation by this element cyclically permutes the factors in (4.1). Consequently

$$H_A \cong F^k \rtimes C_k = F \wr C_k. \tag{4.3}$$

It remains to distinguish the groups for different k. The group F is torsion-free [6], hence so is F^k. Every finite subgroup of $F^k \rtimes C_k$ injects into the quotient C_k, and therefore has order at most k. The splitting in (4.2) supplies a subgroup of order exactly k. Thus the maximum order of a finite subgroup of H_A is precisely k, an abstract isomorphism invariant. This completes the proof of Theorem 1.1.

Remark 4.1. *For $k = 1$, Theorem 1.1 recovers the familiar maximal point stabilizer $F < T$. The cases $k \geq 2$ give the countably infinite family of new abstract types. For each fixed k, all such stabilizers are conjugate in T; as k varies, they are pairwise nonisomorphic.*

5 Literature and priority boundary

The closest checked results are the structure theory for finite-set stabilizers in F [7, 9], finite same-tail-class stabilizers and type systems in V [4], the maximality of T in V [3], and finite-set stabilizers in ample groups [8]. None states Theorem 1.1 or the circular extension criterion. Targeted searches of arXiv, OpenAlex, zbMATH, Crossref, general web indexes, and the reference lists of these papers through September 4, 2026 located no direct prior statement. Since a negative search cannot prove priority and the argument is elementary once formulated, we describe the result only as *apparently unrecorded* and make no absolute priority claim.

The theorem concerns the classical orientation-preserving group T and finite subsets of its dyadic orbit. It does not classify all maximal subgroups of T and does not address stabilizers of arbitrary nondyadic points.

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