Maximal Generic Iterated Galois Images Do Not Determine p-Adic Julia Sets

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September 2, 2026

Abstract

An AIM problem asks whether the arithmetic or geometric Galois group of the generic iterated-preimage tower of a rational map over a p-adic field determines its Julia set. We give a uniform negative answer. For every prime p, the quadratic polynomials

$$z^2 + 1 \qquad \text{and} \qquad z^2 - p^{-6}$$

over $\mathbb{Q}_p$ have arithmetic and geometric generic iterated Galois images equal to the full group $\mathrm{Aut}(T_2)$. The first map has good reduction: its Berkovich Julia set is the Gauss point and its classical Julia set is empty. The second has a type-I Cantor Julia set; moreover, its two contracting inverse branches are defined over $\mathbb{Q}_p$, so the classical Cantor set already lies in $\mathbb{P}^1(\mathbb{Q}_p)$. Thus even the full rooted-tree action and the marked normal pair of arithmetic and geometric images fail to determine the cardinality or topological type of the Julia set. The result combines Pink's maximality theorem with a base-field inverse-branch construction. This note is unrefereed and makes no absolute priority claim.

1 Introduction

Let $k/\mathbb{Q}_p$ be finite and let $f \in k(z)$ have degree $d \geq 2$. For an indeterminate t, the roots of $f^n(z) = t$, linked by f, form the regular rooted d-ary tree. Adjoining all such roots produces a Galois tower whose arithmetic and geometric images will be denoted by $G_\infty(f)$ and $G_\infty^{\mathrm{geom}}(f)$. An AIM problem asks whether the p-adic Julia set of f can be reconstructed from either group [AIM].

There is an important distinction between this profinite image and the discrete iterated monodromy invariant used in complex dynamics. In expanding settings, a discrete iterated monodromy group together with its self-similar correspondence encodes the Julia dynamics [Nek03]. The bare profinite closure in the AIM question can forget substantially more. Pink proved that every quadratic polynomial with infinite postcritical orbit has both generic images equal to the full binary-tree automorphism group [Pin13, Theorem 1.10.2]. Consequently, any two such polynomials have the same profinite tree image, regardless of their non-Archimedean metric geometry.

The point of this note is to make that information loss explicit and uniform in p, while avoiding an ambiguity about the ground field of the classical Julia set. The parameter $-p^{-6}$ is chosen so that the two inverse branches of the escaping quadratic are defined over $\mathbb{Q}_p$ even when $p = 2$.

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2 Generic iterated Galois images

Fix a separable closure of $k(t)$. Let $K_n(f)$ be the splitting field of $f^n(z) - t$ and put

$$K_\infty(f) = \bigcup_{n\geq 0} K_n(f), \qquad L(f) = K_\infty(f) \cap \overline{k}.$$

We use

$$G_\infty(f) = \mathrm{Gal}(K_\infty(f)/k(t)), \qquad G_\infty^{\mathrm{geom}}(f) = \mathrm{Gal}(K_\infty(f)/L(f)(t)). \tag{2.1}$$

After labelling the generic preimage tree, both are closed subgroups of $\mathrm{Aut}(T_d)$, well defined up to conjugacy.

For a quadratic polynomial in characteristic different from two, the two critical points are its finite critical point and infinity. The latter is fixed. Pink's infinite polynomial theorem may therefore be stated in the form needed here.

Theorem 2.1 (Pink's maximality theorem)**.** *Let f be a quadratic polynomial over a field of characteristic different from two. If its finite critical point has infinite forward orbit, then*

$$G_\infty^{\mathrm{geom}}(f) = G_\infty(f) = \mathrm{Aut}(T_2).$$

This is [Pin13, Theorem 1.10.2].

The characteristic hypothesis concerns the field, not its residue field; thus theorem 2.1 applies over $\mathbb{Q}_2$.

3 The counterexample

Theorem 3.1. *For every prime p, set*

$$f_0(z) = z^2 + 1, \qquad f_1(z) = z^2 - p^{-6} \quad \textit{in } \mathbb{Q}_p[z]. \tag{3.1}$$

Then

$$G_\infty(f_0) = G_\infty^{\mathrm{geom}}(f_0) = \mathrm{Aut}(T_2) = G_\infty(f_1) = G_\infty^{\mathrm{geom}}(f_1). \tag{3.2}$$

On the other hand, the Berkovich Julia sets over $\mathbb{C}_p$ are respectively the singleton Gauss point and a Cantor set of type-I points. The classical Julia sets in $\mathbb{C}_p$ are respectively empty and Cantor. The same distinction already holds on $\mathbb{P}^1(\mathbb{Q}_p)$: the first classical Julia set is empty, while the second is a Cantor set.

Proof. The finite critical orbit of f_0 is the integer sequence

$$a_0 = 0, \qquad a_{n+1} = a_n^2 + 1,$$

which begins $0, 1, 2, 5, 26, 677, \ldots$ and is strictly increasing after a_0. Its terms remain distinct under the injection $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$.

For f_1, write $c = -p^{-6}$ and let $b_0 = 0$, $b_{n+1} = b_n^2 + c$. Since $|c|_p = p^6 > 1$, induction with the ultrametric inequality gives

$$|b_n|_p = |c|_p^{2^{n-1}} \qquad (n \geq 1). \tag{3.3}$$

Indeed, once $n \geq 1$, the square term has norm strictly larger than $|c|_p$. The orbit is therefore infinite. Applying theorem 2.1 twice proves (3.2).

The polynomial f_0 has good reduction at every p: its coefficients are integral, its leading coefficient is a unit, and the reduced map retains degree two. The good-reduction theorem gives

$$J_{\mathrm{Berk}}(f_0) = \{\zeta_{0,1}\}, \tag{3.4}$$

where $\zeta_{0,1}$ is the type-II Gauss point [Ben19, Chapter 8]. Hence the classical Julia set is empty.

For odd p, the non-Archimedean quadratic classification says that $|c|_p > 1$ gives a type-I Cantor Julia set. In residue characteristic two, the corresponding threshold is $|c|_2 > 4$; here $|c|_2 = 64$ [DKY22, Sections 5.1 and 6.1]. Thus $J_{\mathrm{Berk}}(f_1)$ is a type-I Cantor set for every p. The fact that this Cantor set is already defined over $\mathbb{Q}_p$ is proved directly in lemma 4.1. This establishes every asserted Julia-set comparison. □

4 The Cantor coding over $\mathbb{Q}_p$

Lemma 4.1. *Let $a = p^{-3}$, $R = |a|_p = p^3$, and $f(z) = z^2 - a^2$. The classical Julia set of f in $\mathbb{P}^1(\mathbb{Q}_p)$ is homeomorphic to $\{-,+\}^{\mathbb{N}}$, and f on that set is conjugate to the one-sided shift.*

Proof. Let $D = D(0, R)$. If $y \in D$, then a solution of $f(z) = y$ has the form

$$z = \pm a\sqrt{1 + y/a^2}, \qquad |y/a^2|_p \le p^{-3}. \tag{4.1}$$

For odd p, every element of $1 + p\mathbb{Z}_p$ has a square root in $\mathbb{Q}_p$. For $p = 2$, the right side of (4.1) lies in $1 + 8\mathbb{Z}_2$, again a square. Both inverse branches are consequently defined over $\mathbb{Q}_p$.

Put $\rho = |2|_p^{-1}$. Since $R > |2|_p^{-2}$, expansion at the two roots gives the disjoint decomposition

$$f^{-1}(D) = D(a, \rho) \,\dot\cup\, D(-a, \rho). \tag{4.2}$$

Each disk in (4.2) maps bijectively onto D. For points z, z' in one of the two disks,

$$|z - z'|_p = \frac{|f(z) - f(z')|_p}{|z + z'|_p} = \frac{|f(z) - f(z')|_p}{|2|_p R}. \tag{4.3}$$

Thus either inverse branch contracts by

$$q = (|2|_p R)^{-1} < 1; \tag{4.4}$$

at $p = 2$, $q = 1/4$.

For every sign word of length n, iterating the corresponding inverse branches gives one closed disk, and the 2^n disks are pairwise disjoint. Their diameters tend to zero by (4.4). Completeness and local compactness therefore identify

$$K = \bigcap_{n \ge 0} f^{-n}(D) \tag{4.5}$$

homeomorphically with $\{-,+\}^{\mathbb{N}}$, and $f|_K$ shifts the coding.

Every point outside K eventually leaves D and then escapes locally uniformly to infinity. Periodic sign sequences are dense in K. Their points are repelling, since a period-n multiplier has norm

$$\prod_{j=0}^{n-1} |2f^j(z)|_p = (|2|_p R)^n > 1. \tag{4.6}$$

Hence K is contained in the classical Julia set, while its complement is Fatou. Therefore $J_{\mathbb{Q}_p}(f) = K$. □

The same inverse disks over $\mathbb{C}_p$ give the standard type-I Cantor Julia set. The local-field shift alternative is also developed in [BBP07].

5 What the profinite image forgets

For the monic centered family $f_c(z) = z^2 + c$, the first generic cover is

$$z^2 + c - t = 0, \qquad \mathrm{disc}_z(z^2 + c - t) = 4(t - c). \tag{5.1}$$

Thus the full cover over the valued, coordinate-labelled t-line remembers the finite branch place $t = c$ and in particular $v_p(c)$. The two maps in theorem 3.1 have branch valuations 0 and -6. Their common image $\mathrm{Aut}(T_2)$ retains no label for that valuation.

This diagnoses the loss for the displayed family, but does not prove a positive reconstruction theorem from the whole valued extension. Such an enrichment may need labelled branch places and compatible inertia and decomposition subgroups throughout the tower. Nor do we claim that (5.1) is a universally minimal repair.

Remark 5.1 (Abstract group versus tree action). The counterexample does not rely on forgetting the permutation action. Once either generic preimage tree is labelled, its image is literally the whole closed subgroup $\mathrm{Aut}(T_2)$, with its standard action. The marked inclusion $G_\infty^{\mathrm{geom}} \triangleleft G_\infty$ is also the equality $\mathrm{Aut}(T_2) \triangleleft \mathrm{Aut}(T_2)$ for both maps.

Remark 5.2 (Coordinate conjugacy). Projectively conjugate maps already obstruct recovery as an embedded subset without a coordinate normalization. Here the obstruction is stronger: a singleton and a Cantor set are not homeomorphic, so no projective conjugacy can relate the two Julia sets.

6 Verification and literature boundary

Two exact standard-library checks accompany this note. One propagates the critical valuations, checks the odd and dyadic Julia thresholds, and enumerates the finite groups generated by Pink's canonical inertia recursion through tree height four, obtaining orders

$$2, \quad 8, \quad 128, \quad 32768 = 2^{2^n - 1} \quad (n = 1, 2, 3, 4).$$

The second implementation instead checks the independent sign matrices and conjugacy orbits through height nine and recomputes the base-field square-root conditions with exact rational arithmetic. These computations verify the explicit hypotheses; the cited infinite theorems remain the mathematical inputs.

The closest positive reconstruction statements located are for richer discrete self-similar invariants [Nek03]. Pink supplies the maximal profinite group but does not compare p-adic Julia sets; the local quadratic sources [BBP07, DKY22] supply the Julia regimes but do not make the Galois comparison. Jones relates iterated Galois towers to the p-adic Mandelbrot set [Jon07], without stating the present non-reconstruction pair. A targeted search through 29 August 2026 located no source stating theorem 3.1. This negative search is not a proof of priority, and no "first" claim is made.

AI-assisted tools were used to enumerate the frozen problem corpus, locate references, test the exact finite-level calculations, and help prepare this manuscript. The submitting author must inspect the proof and sources, assume responsibility for every claim, and comply with the applicable submission disclosure policy. The note is not peer reviewed.

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