

Minimum-Volume Central Slabs of the Regular Octahedron

Alper Ferudun*

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Abstract

We determine the minimum volume of the intersection of a regular octahedron with a centered slab of every prescribed width, including all equality directions. For the octahedron with vertices $\pm e_i$ and slab half-width t , the minimizing normal is diagonal below a single threshold $\tau = 0.4740349423895888\dots$ and is a coordinate normal above it, until the slab contains the whole body. The threshold is given exactly by a quadratic over $\mathbb{Q}(\sqrt{3})$. The proof derives explicit chamber formulas, reduces the optimization to normals with two equal coordinates, and excludes interior extrema by sign-monotone derivatives and a polynomial resultant. This answers the three-dimensional minimum-direction part of an AIM question on cross-polytope slabs; higher dimensions and maximum volumes are not addressed. No absolute priority or independent peer-review claim is made.

1 Statement and scope

Let

$$B = \{x \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| \leq 1\}, \quad V(w, t) = \frac{\text{vol}_3\{x \in B : |\langle w, x \rangle| \leq t\}}{4/3},$$

where $\|w\|_2 = 1$ and $t \geq 0$. Thus $2t$ is the physical width of the slab and $4/3$ is the volume of B . For $j = 1, 2, 3$, write f_j for the unit vector with j equal positive nonzero coordinates.

Theorem 1.1. *Put $\kappa = 3\sqrt{3}/8$, and let τ be the unique root in $(0, 1/\sqrt{3})$ of*

$$(1 + \kappa)\tau^2 - 3\tau + 3 - 3\kappa = 0. \tag{1}$$

Then

$$\min_{\|w\|_2=1} V(w, t) = \begin{cases} \kappa t(3 - t^2), & 0 \leq t \leq \tau, \\ 1 - (1 - t)^3, & \tau \leq t \leq 1, \\ 1, & t \geq 1. \end{cases} \tag{2}$$

For $0 < t < \tau$, precisely the signed permutations of f_3 minimize. For $\tau < t < 1$, precisely the signed coordinate normals minimize. At $t = \tau$, these two classes and no other normals minimize. Every normal minimizes at $t = 0$ and at $t \geq 1$. The transition is

$$\tau = \frac{96 - 36\sqrt{3} - 8\sqrt{33} + 9\sqrt{11}}{37} = 0.47403494238958883233\dots$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>. Unrefereed preprint. CC BY 4.0.

Problem 23(d) in the AIM list [1] asks for formulas for cross-polytope slabs and noncentral sections and for extremal slab directions in general dimension. Theorem 1.1 concerns only the minimum in dimension three. It does not settle maximum slabs, higher-dimensional extrema, or parts (a)–(c) of that problem, which concern cubes and negative spherical moments.

Liu and Tkocz [2, Theorem 5] prove the coordinate minimum for cross-polytopes of dimension $n \geq 3$ when $1/\sqrt{2} < t \leq 1$. Our result agrees with that range in dimension three and determines the complete smaller-width transition. We explicitly separate $t = 1$, when every slab contains the entire body, from the uniqueness statement. König [3] treats noncentral hyperplane sections, local extrema and perimeters. The fixed-offset section functional is different from its integral, the slab-volume functional. The simplex formula used below is standard; neither it nor the previously known wide-slab result is claimed as new. A bounded literature search is not a certificate of absolute priority.

2 Exact volume formulas

Signed coordinate symmetry allows us to write $w = (a, b, c)$ with $a \geq b \geq c \geq 0$ and $a^2 + b^2 + c^2 = 1$. Set

$$q = ab + ac + bc, \quad D = (a + b)(a + c)(b + c), \quad T = 1 - V.$$

For distinct positive coordinates, the normalized omitted volume is

$$T(a, b, c; t) = \sum_{x \in \{a, b, c\}} \frac{x(x - t)_+^3}{\prod_{y \in \{a, b, c\} \setminus \{x\}} (x^2 - y^2)}, \quad u_+ = \max\{u, 0\}. \quad (3)$$

To see the normalization, divide B into its eight signed coordinate simplices. On a simplex, an affine functional with distinct vertex values $d_0, \dots, d_3$ has normalized tail

$$\sum_{i=0}^3 \frac{(d_i - t)_+^3}{\prod_{j \neq i} (d_i - d_j)};$$

an elementary proof is given in the appendix. Here $d_0 = 0$. Since $t \geq 0$, only positive knots contribute. The contribution of the knot a_i , summed over signs on the other two axes, has coefficient

$$\frac{1}{a_i} \prod_{j \neq i} \left(\frac{1}{a_i - a_j} + \frac{1}{a_i + a_j} \right) = \frac{4a_i}{\prod_{j \neq i} (a_i^2 - a_j^2)}.$$

The eight equal simplex weights and the two symmetric tails give (3). Volume is continuous in (w, t) by dominated convergence: the boundary hyperplanes have three-dimensional measure zero. Therefore zero and repeated coordinates are limits of the full expression, not deletions of repeated knots.

Simplifying (3) gives the following stable chamber formulas:

$$V = \begin{cases} \frac{t(3q - t^2)}{D}, & 0 \leq t \leq c, \\ \frac{t(3q - t^2)}{D} - \frac{c(t - c)^3}{(a^2 - c^2)(b^2 - c^2)}, & c < t < b, \\ 1 - \frac{a(a - t)^3}{(a^2 - b^2)(a^2 - c^2)}, & b \leq t < a, \\ 1, & t \geq a. \end{cases} \quad (4)$$

Degenerate endpoints are interpreted by continuity. All denominators are positive in the nonempty open chambers. At $t = b$ or $t = c$ with distinct adjacent coordinates, the first derivatives agree: the term entering or leaving (3) vanishes to third order.

3 The chamber with three positive tail terms

Lemma 3.1. *If $0 < t \leq c$, then $V(a, b, c; t) \geq V(f_3, t)$, with equality only when $a = b = c$.*

Proof. First fix c , and put $s = a + b$ and $p = ab = (s^2 + c^2 - 1)/2$. The first formula in (4) becomes

$$V_0 = \frac{t(3p + 3cs - t^2)}{s(p + cs + c^2)}, \quad (V_0)_s = -\frac{tP}{4D^2},$$

where

$$\frac{P}{4} = 3a^2b^2 + 6a^2bc + 6ab^2c + 3abc^2 - t^2(a^2 + 3ab + 2ac + b^2 + 2bc + c^2).$$

The coefficient of $-t^2$ is positive, so $P(t) \geq P(c)$. Moreover,

$$\frac{P(c)}{4} = 3a^2b^2 + 6a^2bc - a^2c^2 + 6ab^2c - 2ac^3 - b^2c^2 - 2bc^3 - c^4 > 0.$$

Indeed, $3a^2b^2 - a^2c^2 \geq 2a^2b^2$ and $6a^2bc - 2ac^3 \geq 4a^2bc$. The other negative terms have total absolute value at most $4ab^2c$, leaving at least $2ab^2c$ from $6ab^2c$. We used $a, b \geq c$. Increasing s at fixed c therefore strictly decreases V until $a = b$.

Now set $a = b = (2 + r^2)^{-1/2}$ and $c = r(2 + r^2)^{-1/2}$, where $0 < r \leq 1$. Direct differentiation gives

$$\frac{dV_0}{dr} = -\frac{t(r-1)\{t^2(r^3 + 4r^2 + 2r + 8) - 9r\}}{2(r+1)^3\sqrt{r^2+2}}.$$

Since $t \leq c$, the brace is at most

$$\frac{r}{r^2+2}(r^4 + 4r^3 - 7r^2 + 8r - 18) < 0;$$

the positive terms satisfy $r^4 + 4r^3 + 8r \leq 13 < 18$. Thus increasing $r < 1$ strictly decreases V_0 , while preserving $t \leq c$. The endpoint is f_3 , proving the claim. $\square$

4 Reduction to two equal coordinates

Fix c . The ordered arc of (a, b) can be parametrized by $s = a + b$, from $\sqrt{1 - 2c^2} + c$ to $\sqrt{2(1 - c^2)}$. Its endpoints have $b = c$ and $a = b$, respectively. In the open arc a decreases and b increases with s . We prove that no interior point with $V < 1$ can be a minimum. Lemma 3.1 and its fixed- c calculation cover $t \leq c$, so suppose $t > c$.

4.1 One positive tail term

If $b < t < a$, write $k = c^2$. Then

$$T = \frac{a(a-t)^3}{(2a^2 + k - 1)(a^2 - k)}.$$

Define

$$F = 6a^4 - (1+k)a^2 - k(1-k), \quad H(a, k) = \frac{2a\{(1+k)a^2 - 2k(1-k)\}}{F}.$$

Differentiation gives

$$T_a = \frac{(a-t)^2 F \{t - H(a, k)\}}{(a^2 - k)^2 (2a^2 + k - 1)^2}. \quad (5)$$

Here $0 \leq k < 1/3$ and $a^2 \geq (1-k)/2$. The polynomial F is positive: its value at the left endpoint is $(1-k)(1-3k) > 0$, and it increases thereafter. The function H strictly decreases with a . For an explicit verification, the numerator of H_a over its positive denominator F^2 , after setting $z = a^2 - (1-k)/2$, is

$$\begin{aligned} & -12(k+1)z^3 - 4(7k-5)(2k-1)z^2 \\ & + (3k-1)(k-1)(25k-11)z - 2(1-k)^2(1-3k)^2. \end{aligned}$$

All coefficients are nonpositive for $0 \leq k \leq 1/3$, and the constant is negative for $k < 1/3$. As s increases, a decreases and H increases. Since $V_s = -T_a a_s$ with $a_s < 0$, the sign of V_s is that of $t - H$. It can only pass from positive to negative, so no interior local minimum occurs in this chamber.

4.2 Two positive tail terms

Suppose $c < t < b$. Keep $p = (s^2 + c^2 - 1)/2 = ab$, and put

$$E = (a^2 - c^2)(b^2 - c^2) = p^2 - c^2(1 - 2c^2), \quad W = -\frac{E^2 V_s}{tps}.$$

Since $V = V_0 - c(t-c)^3/E$ and $E_s = 2ps$, the second term in W is the constant $-2c(t-c)^3/t$. Differentiating the other term gives

$$W_s = \frac{EL}{a^2 b^2 (a+b)^4}, \quad (6)$$

where

$$\begin{aligned} B_0 &= a^4 + 5a^3b + 5a^2b^2 + 5ab^3 + b^4 - c^2(a^2 + 5ab + b^2), \\ L &= 9a^2b^2(a^2 + ab + b^2 - c^2) - t^2B_0. \end{aligned}$$

This derivative is strictly positive. First, $B_0 \geq B_0|_{c=b} = a^2(a+b)(a+4b) > 0$, so $t \leq b$ implies $L \geq L|_{t=b}$. The coefficient of c^2 in this last polynomial is $-b^2(8a^2 - 5ab - b^2) < 0$, because $a \geq b > 0$. Hence

$$L \geq L|_{t=b, c=b} = 4a^2b^2(a+b)(2a-b) > 0.$$

The sign of V_s is that of $-W$, so it can again only pass from positive to negative. There is no interior local minimum.

4.3 Chamber junctions

As s increases, the only transition between the two preceding open chambers is $b = t$. Their first derivatives agree there. If this common derivative is zero, the sign ordering just proved gives a nonnegative derivative on its left and a nonpositive derivative on its right. Thus no minimum appears at the junction. Regions with $t \geq a$ have volume one and cannot contain a global minimum for $0 < t < 1$, since a coordinate slab has volume strictly below one. Strictness also excludes a constant subarc at a value below one. We have proved:

Proposition 4.1. *For $0 < t < 1$, every minimizing normal has $a = b$ or $b = c$ after signed permutation.*

5 The equal-coordinate families

5.1 Two equal smaller coordinates

Let $b = c$. Lemma 3.1 covers $t \leq c$. In the remaining nontrivial chamber $c < t < a$, we have $c^2 = (1 - a^2)/2$ and

$$T = \frac{4a(a-t)^3}{(3a^2-1)^2}, \quad T_a = \frac{4(a-t)^2\{t(9a^2+1)-4a\}}{(3a^2-1)^3}. \quad (7)$$

If $0 < t < 1/\sqrt{3}$, the smaller root of the quadratic in braces is $t/(2 + \sqrt{4 - 9t^2}) < 1/\sqrt{3}$. For $a \geq 1/\sqrt{3}$ its sign can only change from negative to positive, excluding an interior maximum of T . The endpoints are $a = 1$ and $c = t$; the latter is covered by Lemma 3.1.

If $t \geq 1/\sqrt{3}$, then $a > t$ in the nontrivial interval. The brace in (7) is nonnegative at $a = t$, and its derivative for $a \geq t$ is $18ta - 4 \geq 2$. Thus the largest tail is at $a = 1$. This family gives no volume below $\min\{V(f_1, t), V(f_3, t)\}$ and no additional equality case.

5.2 Two equal larger coordinates

Let $a = b$. Lemma 3.1 covers $t \leq c$, and $t \geq a$ gives volume one. If $c < t < a$, then $c^2 = 1 - 2a^2$, and the repeated-knot limit of (3) is

$$T = \frac{(a-t)^2\{10a^3 - 4a + (1-a^2)t\}}{2a(3a^2-1)^2}. \quad (8)$$

For example, this limit is the derivative at $x = a^2$ of $\sqrt{x}(\sqrt{x} - t)^3/(x - c^2)$, with c fixed before substituting $c^2 = 1 - 2a^2$.

Put $u = a^2$, $r = t/a$. In the closure of the nontrivial chamber,

$$0 < r < 1, \quad \frac{1}{2+r^2} \leq u \leq \frac{1}{2}.$$

Define

$$Q(u, r) = u^2(9r^2 - 63r) + u(16 + 34r - 14r^2) + r^2 + r - 8.$$

For a fixed value of t , differentiation of (8) gives

$$T_a = \frac{(t-a)Q(u, r)}{2(3a^2-1)^3}. \quad (9)$$

We show that every interior stationary point of T is a strict local minimum. Write $J = 2uQ_u - rQ_r$. As a function of u , Q is a strictly concave quadratic, with endpoint values

$$Q\left(\frac{1}{2+r^2}, r\right) = \frac{r(r-1)^3(r^2+4r-9)}{(r^2+2)^2} > 0, \quad (10)$$

$$Q(1/2, r) = -\frac{3r(5r-3)}{4}. \quad (11)$$

There is no interior root for $0 < r \leq 3/5$. For $3/5 < r < 1$, there is exactly one: a concave quadratic starting positive can cross zero only once before a negative right endpoint. Its u -derivative at that crossing is nonzero. Consequently these roots form one continuous curve, with endpoint $(u, r) = (1/2, 3/5)$.

An explicit elimination identity is

$$\text{Res}_u(Q, J) = -576r(r-1)^3(10r^4 - 116r^3 + 129r^2 + 329r - 448). \quad (12)$$

The quartic factor is negative on $[0, 1]$. Its derivative is $40r^3 + 329 + 258r - 348r^2 > 0$, since the concave quadratic part is at least 239 on this interval, and the quartic equals -96 at $r = 1$. Thus the resultant does not vanish for $0 < r < 1$, and J cannot vanish on the critical curve. At its endpoint $J(1/2, 3/5) = -37/20$, so $J < 0$ on the entire curve.

Along a fixed- t variation, $dQ/da = J/a$. At a stationary point, (9) therefore gives $T_{aa} > 0$, because $t - a < 0$. Hence there is no interior maximum tail. The endpoints of a closed admissible interval are $c = 0$, giving f_2 ; $c = t$, covered by Lemma 3.1; or $a = t$, giving zero tail. Degenerate endpoints follow by continuity. This family gives no volume below $\min\{V(f_2, t), V(f_3, t)\}$.

Together with Proposition 4.1, this proves that one of f_1, f_2, f_3 attains the minimum, and that no additional direction minimizes at a value below one.

6 The transition and equality cases

The three remaining normalized volumes are

$$\begin{aligned} C(t) &= 1 - (1-t)^3 & (0 \leq t \leq 1), \\ G_2(t) &= (3t - 2t^3)/\sqrt{2} & (0 \leq t \leq 1/\sqrt{2}), \\ G_3(t) &= \kappa t(3 - t^2) & (0 \leq t \leq 1/\sqrt{3}), \end{aligned}$$

with each function equal to one beyond its displayed endpoint.

The two-coordinate direction never minimizes for $0 < t < 1$. For an exact comparison, set $\rho = 19/40$. On $0 < t \leq \rho$, the quotient $(G_2 - G_3)/t$ decreases with t^2 and at ρ equals

$$\frac{16312\sqrt{2} - 13317\sqrt{3}}{12800} > 0, \quad 2(16312)^2 - 3(13317)^2 = 135221 > 0.$$

On $[\rho, 1/\sqrt{2}]$, the quotient $(G_2 - C)/t$ is a concave quadratic, positive at both endpoints. At ρ it is

$$\frac{2039\sqrt{2} - 2881}{1600} > 0, \quad 2(2039)^2 - (2881)^2 = 14881 > 0;$$

at $1/\sqrt{2}$ we have $G_2 = 1 > C$. The same last inequality holds for $1/\sqrt{2} \leq t < 1$.

Finally, on $[0, 1/\sqrt{3}]$,

$$C(t) - G_3(t) = t\{(1 + \kappa)t^2 - 3t + 3 - 3\kappa\}.$$

The bracket strictly decreases, is positive at zero and negative at $1/\sqrt{3}$. It therefore has exactly the root (1) in that interval. Beyond $1/\sqrt{3}$ the diagonal volume is one, so the coordinate direction is better until $t = 1$. This proves (2), the radical expression for τ , and, using the strict reductions above, all equality cases in Theorem 1.1.

The argument is analytic. The accompanying SymPy script checks 31 identities, the resultant, boundary comparisons and exact root isolation. Separate floating-point clipping tests agree with the chamber formulas, but no numerical sample or optimization output is used to prove the theorem. Unsuccessful real-algebraic solver attempts are retained in the research archive and are not certificates. AI tools assisted the author with algebra, source searches, exposition and code checks. This is a self-audited, unrefereed preprint, not an independent referee report or proof-assistant formalization; the author is responsible for its claims.

A The elementary simplex tail formula

For completeness, let a point be uniformly distributed on an n -simplex, and let an affine functional take distinct values $d_0, \dots, d_n$ at its vertices. Its tail is

$$\mathbb{P}(Y > t) = \sum_{i=0}^n \frac{(d_i - t)_+^n}{\prod_{j \neq i} (d_i - d_j)}. \quad (13)$$

In dimension one this is the length of an interval truncated at t . For the inductive step, condition on the last barycentric coordinate s , which has density $n(1-s)^{n-1}$ on $[0, 1]$. The other normalized coordinates are uniform on the opposite $(n-1)$ -simplex. Inserting the inductive formula and integrating, each term with $i < n$ becomes

$$\frac{(d_n - t)_+^n - (d_i - t)_+^n}{(d_n - d_i) \prod_{j < n, j \neq i} (d_i - d_j)}.$$

The coefficient of $(d_i - t)_+^n$ is the one in (13). For the coefficient of $(d_n - t)_+^n$, the partial-fraction identity

$$\sum_{i < n} \frac{1}{(z - d_i) \prod_{j < n, j \neq i} (d_i - d_j)} = \frac{1}{\prod_{j < n} (z - d_j)}$$

at $z = d_n$ gives the remaining term. This proves (13). Repeated knots follow by continuity of the uniform integral.

References

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