

Minimum Perimeter of Central Hyperplane Sections of the Cube

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Abstract

For every $n \geq 3$, we prove that a central hyperplane section of the side-one real cube has perimeter at least $2(n-1)$, where perimeter is the $(n-2)$ -dimensional measure of its relative boundary. Equality holds exactly for coordinate sections. The proof is analytic in dimensions three and four and in every dimension at least seven; dimensions five and six use complete finite certificates checked with exact arithmetic. The analytic argument orders the coupled facet densities and combines this ordering with a cubic negative-moment estimate for section volume. A quantitative comparison of nearby graph sections turns exact center calculations into rigorous bounds on entire cells. This answers the minimum-perimeter part of an AIM problem; other functionals, codimensions and measures in the bundled source question are not addressed. No absolute priority or independent peer-review claim is made.

1 Statement and scope

Write $Q_n = [-1/2, 1/2]^n$. For a unit vector $u \in \mathbb{R}^n$, let

$$K_n(u) = Q_n \cap u^\perp, \quad A_n(u) = \mathcal{H}^{n-1}(K_n(u)), \quad P_n(u) = \mathcal{H}^{n-2}(\partial_{u^\perp} K_n(u)).$$

The boundary is taken in the section hyperplane. We also write $P_n(v)$ and $A_n(v)$ for a nonzero, not necessarily unit, normal, since its scale does not change the section.

Theorem 1.1. *For every $n \geq 3$ and every unit vector $u \in \mathbb{R}^n$,*

$$P_n(u) \geq 2(n-1).$$

Equality holds if and only if $u = \pm e_j$ for some coordinate vector e_j .

For the side-two cube $[-1, 1]^n$, the minimum is consequently $2^{n-1}(n-1)$. In dimension two the zero-dimensional boundary measure of every central line section is two; uniqueness is restricted to $n \geq 3$.

König's item 2 in Problem 1 of the 2013 AIM list [1] asks for the minimum in the side-two normalization. Theorem 1.1 answers that item. It does not resolve the source's other questions about mean width, intrinsic volumes, codimensions, Gaussian measures or projections. König and Koldobsky [2] proved the maximum theorem, derived facet identities and explicit low-dimensional

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formulas, and recorded the conjectured coordinate minimum together with a weaker lower bound. We credit those identities and formulas throughout.

The probabilistic part of the proof uses the sphere representation and moment method in Chasapis, Nayar and Tkocz [3, Lemma 12 and Section 6.1]. The additional ingredients are the ordered facet comparison, the explicit perimeter estimates, and the finite certificates. Our targeted literature search is not a certificate of absolute priority. This is a computer-assisted, unrefereed argument, not a proof-assistant formalization.

2 Comparison of nearby graph sections

We first give a quantitative continuity estimate needed for both the boundary cases and the finite computations. Surface area is monotone under inclusion of full-dimensional convex bodies in a fixed Euclidean space. For polytopes, this follows directly from the Cauchy projection formula: for a unit vector z , the volume of the projection perpendicular to z is

$$\frac{1}{2} \sum_F |F| |\nu_F \cdot z|,$$

where F ranges over facets and ν_F is its unit normal. Integrating over z gives a positive constant times surface area, whereas each projection volume is monotone under inclusion. See also [4, Theorem 2].

Lemma 2.1. *For $b, c \in \mathbb{R}^{n-1}$ and $n \geq 3$,*

$$P_n(1, b) \geq \frac{P_n(1, c)}{(1 + \|b - c\|_2)(1 + \|b - c\|_1)^{n-2}}. \quad (1)$$

In particular, P_n is continuous on the space of nonzero normal directions, including directions with zero coordinates.

Proof. Put $d = n - 1$ and define

$$T_b y = (-b \cdot y, y), \quad D_b = \{y \in Q_d : |b \cdot y| \leq 1/2\}, \quad K_b = T_b D_b.$$

For $\gamma = (1 + \|b - c\|_1)^{-1}$, the inequality $|b \cdot y| \leq (1 + \|b - c\|_1)/2$ on D_c gives $\gamma T_b D_c \subseteq K_b$. Consider the isomorphism $L = T_b T_c^{-1} : \text{im } T_c \rightarrow \text{im } T_b$. Its least singular value is at least $(1 + \|b - c\|_2)^{-1}$, since

$$|T_c y| \leq |T_b y| + \|b - c\|_2 |y| \leq (1 + \|b - c\|_2) |T_b y|.$$

Moreover, at most one singular value of L is below one. Its squared singular values are the generalized eigenvalues of $I + bb^T$ relative to $I + cc^T$. The difference $bb^T - cc^T$ has at most one negative eigenvalue; congruence by $(I + cc^T)^{-1/2}$ preserves this inertia. Thus the product of the smallest $d - 1$ singular values is at least $(1 + \|b - c\|_2)^{-1}$. Every facet area of LK_c is at least this factor times the original area. Surface-area monotonicity, applied to $\gamma LK_c \subseteq K_b$, proves (1).

Every D_b contains a neighborhood of the origin. Reversing b, c in (1) proves continuity in this coordinate chart. At any nonzero normal one can choose a nonzero coordinate for the chart, giving the assertion. $\square$

3 Elementary regions and dimensions three and four

3.1 The dominant-coordinate graph

By signs and permutations we may work with a normal $(a, b_1, \dots, b_d)$, where $a > 0$, $b_i \geq 0$, and $d = n - 1$. Put $S = \sum b_i$, $R^2 = \sum b_i^2$, and $N^2 = a^2 + R^2$. The graph of $-b \cdot y/a$ over Q_d has perimeter

$$P_{\text{box}} = \frac{2}{a} \sum_{i=1}^d \sqrt{N^2 - b_i^2} \geq 2 + 2(d-1)N/a. \quad (2)$$

Indeed its Gram matrix is $I + bb^T/a^2$. For the inequality, use the chord bound for $\sqrt{1+x}$ on $[0, R^2/a^2]$ and $\sum_i (R^2 - b_i^2) = (d-1)R^2$. If $a \geq S$, this graph is the entire section. It follows that $P_n \geq 2d$, with equality exactly when $b = 0$.

3.2 The three-dimensional case

For $a \geq b \geq c > 0$, the case $a \geq b + c$ follows from (2). If $a < b + c$, direct facet lengths give the hexagon formula

$$P_3(a, b, c) = \sum_{\text{cyc}} \frac{b+c-a}{bc} \sqrt{b^2 + c^2}. \quad (3)$$

These low-dimensional formulas are also given in [2]. All coefficients are positive. Since

$$\sum_{\text{cyc}} \frac{(b+c-a)(b+c)}{bc} = 6,$$

we obtain $P_3 \geq 3\sqrt{2} > 4$.

We will also need $A_3 \geq 1$. In the dominant case $A_3 = N/a$. Otherwise put $h = b + c - a$ and $k = h/a \in (0, 1]$. Removing two corner triangles from the graph domain gives

$$A_3 = \frac{N}{a} \left(1 - \frac{h^2}{4bc} \right) \geq \sqrt{1 + (1+k)^2/2} (1 - k/4) = g(k).$$

Here $bc - ah = (a-b)(a-c) \geq 0$ and $b^2 + c^2 \geq (b+c)^2/2$ justify the two factors. The identity

$$(g^2)'(k) = \frac{(k-4)(k-1)(2k+1)}{16} \geq 0 \quad (0 \leq k \leq 1)$$

gives $A_3 \geq \sqrt{3/2} > 1$ in the hexagonal case. Zero-coordinate cases follow from the graph formula or directly as products. In particular, adjoining one zero coordinate gives $P_4(a, b, c, 0) = P_3(a, b, c) + 2A_3(a, b, c)$.

3.3 The first two-corner clipping chamber

Assume $b_i > 0$ and

$$0 \leq t = (S - a)/2 \leq \min_i b_i. \quad (4)$$

Only two opposite corner simplices are removed from the graph domain. Counting the lost old facets and the two new facets gives

$$P = P_{\text{box}} - c_0 \left(\frac{1}{a} \sum_i b_i \sqrt{N^2 - b_i^2} - R \right), \quad c_0 = \frac{2t^{d-1}}{(d-1)! \prod_i b_i}. \quad (5)$$

On a pair of old facets, the removed base measure is $2t^{d-1}/((d-1)!\prod_{j \neq i} b_j)$, with graph Jacobian $\sqrt{N^2 - b_i^2}/a$. The new pair has measure $c_0 R$, proving (5). Boundary cases follow from Lemma 2.1. Expanding $b_i = t + x_i$, $x_i \geq 0$, also gives

$$\prod_i b_i \geq t^{d-1}(S - (d-1)t). \quad (6)$$

Lemma 3.1. *Every normal satisfying (4) has $P_n > 2(n-1)$ for $n \geq 3$. The dominant-coordinate cone has the non-strict bound, with equality only for coordinate normals.*

Proof. The case $d = 2$ is the three-dimensional calculation above. Suppose first $d \geq 5$. The bracket in (5) is at most SN/a , and

$$c_0 S \leq \frac{2d}{(d-1)!}, \quad N/a \geq \sqrt{1 + 1/d}.$$

The coefficient below is positive, so (2) gives

$$P \geq 2 + 2 \left(d - 1 - \frac{d}{(d-1)!} \right) \sqrt{1 + 1/d} > 2d.$$

For the last inequality, use

$$\frac{d}{(d-1)!} < \frac{d-1}{2(d+1)} < (d-1) \left(1 - \sqrt{\frac{d}{d+1}} \right).$$

The first inequality holds at $d = 5$, and the ratio of its right-hand side to left-hand side increases by $d^3/((d-1)(d+2)) > 1$; the second follows by rationalization.

For $d = 4$, let $s = a/S \in [1/2, 1]$. Equation (6) yields $c_0 S \leq 2/(3(3s-1))$, and $N/a \geq \sqrt{1 + 1/(4s^2)}$. Hence

$$P \geq 2 + \left(6 - \frac{2}{3(3s-1)} \right) \sqrt{1 + 1/(4s^2)} > 8. \quad (7)$$

The coefficient is positive. Squaring the last comparison is equivalent to $q_5(s) > 0$, where

$$q_5(s) = -216s^3 + 805s^2 - 540s + 100.$$

Putting $s = (1+x)/2$, $0 \leq x \leq 1$, proves this by the identity

$$q_5(s) = \frac{17}{4}(1-x)^3 + \frac{257}{4}x(1-x)^2 + 236x^2(1-x) + 149x^3.$$

It remains to treat $d = 3$. Set $s = a/S \in [1/3, 1]$, $r = R/S \geq 1/\sqrt{3}$ and $w = N/a = \sqrt{1 + r^2/s^2}$. We have $c_0 S \leq 1/s$ and $c_0 b_i \leq 1$ by (6) and $t \leq b_i$. Write $T_3 = \sum b_i^3$. Applying the chord inequality to the positive coefficients $2 - c_0 b_i$ in (5) gives

$$P - 6 \geq 4(w-1) - c_0 S \left[(1-r) + (w-1) \left(1 - \frac{T_3}{SR^2} \right) \right].$$

The bracket is nonnegative because $r \leq 1$ and $T_3 \leq SR^2$. Cauchy-Schwarz gives $R^4 \leq ST_3$. It follows that

$$P - 6 \geq \left(4 - \frac{1-r^2}{s} \right) (w-1) - \frac{1-r}{s}. \quad (8)$$

For fixed s the right-hand side increases with r : its first coefficient is positive (at least two) and increasing, as is $w - 1$, and its last term also increases. Substituting $r = 1/\sqrt{3}$, it suffices to show

$$\left(4s - \frac{2}{3}\right) \left(\sqrt{1 + \frac{1}{3s^2}} - 1\right) > \frac{1}{2}.$$

Both sides moved before squaring are positive. The resulting condition is $q_4(s) = -432s^3 + 621s^2 - 192s + 16 > 0$. For $s = (1 + 2x)/3$,

$$q_4(s) = 5(1 - x)^3 + 67x(1 - x)^2 + 203x^2(1 - x) + 13x^3 > 0.$$

Equation (8) therefore gives $P > 6 + (1/\sqrt{3} - 1/2)/s > 6$. $\square$

3.4 Completion of dimension four

Order the normal as $a \geq b \geq c \geq e \geq 0$. The dominant cone and the first clipping chamber leave only $0 < e < h = b + c - a \leq c$. With $M = a^2 + b^2 + c^2$, direct facet integration gives

$$P_4(a, b, c, e) = \sum_{\text{cyc}} \frac{b + c - a}{bc} \sqrt{b^2 + c^2 + e^2} + \frac{2\sqrt{M}}{a} \left(1 - \frac{h^2 + e^2}{4bc}\right). \quad (9)$$

Here the cyclic sum ranges over a, b, c . To derive it, the density of $bU + cV$, for independent uniforms on $[-1/2, 1/2]$, in its outer linear region is $(b + c - 2|z|)/(2bc)$. Averaging at offset $a/2$ over eW stays in this region when $e \leq h$, and gives $h/(2bc)$. Coarea and the opposite facet supply the first cyclic term; the other two are identical calculations. On the pair with the fourth coordinate fixed, the graph domain loses triangles with areas $(h + e)^2/(8bc)$ and $(h - e)^2/(8bc)$, giving the last term.

All three cyclic coefficients are positive. Thus (9) is concave as a function of $e^2 \in [0, h^2]$ and is bounded below by its smaller endpoint value. At $e = 0$ this is $P_3 + 2A_3 > 6$; at $e = h$ Lemma 3.1 gives a value greater than six. Normals with zero coordinates are treated by the graph formula or the product formula above. This proves Theorem 1.1 for $n = 3, 4$, including equality.

4 Ordered facets and a negative-moment bound

Assume initially $u_1 \geq \dots \geq u_n > 0$ and $|u| = 1$. Set $w_i = \sqrt{1 - u_i^2}$ and let C_i be the density at $u_i/2$ of $\sum_{j \neq i} u_j U_j$, where U_j are independent uniforms on $[-1/2, 1/2]$. Coarea and decomposition into pyramids give the familiar facet identities (compare [2, Propositions 2–4])

$$P_n = 2 \sum_i w_i C_i, \quad \sum_i C_i = (n - 1)A_n, \quad C_i \leq A_n. \quad (10)$$

For the second equality, the facet on $x_i = \pm 1/2$ has area $w_i C_i$ and distance $1/(2w_i)$ from the origin. For the last inequality, A_n is the average of the even unimodal remaining-sum density on $[-u_i/2, u_i/2]$, whereas C_i is its endpoint value. Convolution with a uniform preserves even unimodality: on the positive half-line its derivative has the sign of $g(x + h) - g(x - h) \leq 0$.

Lemma 4.1. *For positive ordered coordinates, $C_1 \leq \dots \leq C_n$. Consequently*

$$\frac{P_n}{2} \geq \frac{n - 1}{n} A_n \sum_i \sqrt{1 - u_i^2}. \quad (11)$$

Proof. For $a = u_i \geq b = u_j$, put $\eta = (a - b)/2$ and $t = (a + b)/2$. Let g be the even nonincreasing density from all other coordinates. With $I = \int_{\eta}^t g$ and $J = \int_0^{\eta} g$, convolution gives $C_i = I/b$ and $C_j = (I + 2J)/a$. Monotonicity gives $\eta I \leq bJ$, which proves $C_i \leq C_j$. If $a = b$ they are equal. We use this argument only for $n \geq 7$, so the remaining density is continuous and there is no one-uniform endpoint convention. The w_i have the same increasing order. Chebyshev's sum inequality and (10) imply (11). $\square$

The sphere representation [3, Lemma 12], attributed there to König and Koldobsky, is

$$A_n(u) = \mathbb{E} \left| \sum_i u_i \xi_i \right|^{-1}, \quad (12)$$

where ξ_i are independent uniform vectors on $S^2 \subset \mathbb{R}^3$. Its normalization can also be seen directly: conditionally on radius R , the first coordinate of the rotationally invariant sum is uniform on $[-R, R]$. Its density at zero is $\mathbb{E}(1/(2R))$, while the same coordinate is a sum of uniforms on $[-1, 1]$. Rescaling and coarea give (12). Equivalently, the density-at-zero limit follows by monotone convergence from $\mathbb{E} \min(1, \varepsilon/R)/(2\varepsilon)$.

Put $X = |\sum_i u_i \xi_i|^2$, $s_2 = \sum_i u_i^4$ and $s_3 = \sum_i u_i^6$. The moment calculation from [3, Section 6.1] gives

$$\mathbb{E}X = 1, \quad \mathbb{E}X^2 = \frac{5}{3} - \frac{2}{3}s_2, \quad \mathbb{E}X^3 = \frac{35}{9} - \frac{14}{3}s_2 + \frac{16}{9}s_3. \quad (13)$$

Indeed write $X = 1 + Y$ with $Y = 2 \sum_{i < j} u_i u_j \langle \xi_i, \xi_j \rangle$. Pair second moments equal $1/3$, triangular product moments equal $1/9$, and the other terms in $\mathbb{E}Y^3$ vanish by sign symmetry. Thus $\mathbb{E}Y^2 = 2(1 - s_2)/3$ and $\mathbb{E}Y^3 = 8(1 - 3s_2 + 2s_3)/9$.

The fourth derivative of $x^{-1/2}$ is positive, so its degree-three Taylor polynomial at any $c > 0$ is a global lower bound for $x > 0$. Taking expectations and choosing $c = 4/3$ gives

$$A_n \geq \frac{C + \sum_i q(x_i)}{D}, \quad x_i = u_i^2, \quad C = \frac{1295}{27}, \quad D = \frac{2048\sqrt{3}}{81}, \quad q(x) = \frac{14}{3}x^2 - \frac{80}{9}x^3. \quad (14)$$

To check the constants, before this substitution the numerator is $35c^3 - 35c^2 + 21c\mathbb{E}X^2 - 5\mathbb{E}X^3$ and the denominator is $16c^{7/2}$. Taylor's remainder has the same nonnegative sign on both sides of c ; this is not a local approximation.

5 The analytic proof in every dimension at least seven

Write $m = \max_i x_i$. We first prove the strict bound when $m < 1$.

5.1 Small and intermediate largest coordinates

If $m \leq 1/2$, every $q(x_i) \geq 0$. Minimizing a concave sum over the capped probability simplex gives $\sum_i \sqrt{1 - x_i} \geq n - 2 + \sqrt{2}$. At $n = 7$,

$$\frac{C}{D}(5 + \sqrt{2}) - 7 > \frac{249}{10000} > 0. \quad (15)$$

Since $C/D > 1$, the difference increases with n . Equations (11) and (14) prove the result in this range.

Now let $1/2 \leq m \leq 59/100$ and $\delta = 1 - m$. Concentrating the tail squares gives

$$\sum_i \sqrt{1 - x_i} \geq n - 2 + \sqrt{m} + \sqrt{\delta}. \quad (16)$$

For $n \geq 8$, each tail q is nonnegative, and q decreases on $[1/2, 3/5]$. Thus $q(m) \geq q(3/5) = -6/25$ and

$$\frac{C - 6/25}{D} \left(6 + \sqrt{3/5} + \sqrt{2/5} \right) - 8 > \frac{717}{10000} > 0. \quad (17)$$

Here $(C - 6/25)/D > 1$, so the same bound works in all larger dimensions.

For $n = 7$, put $\alpha = 14/3$, $\beta = 80/9$ and $z = \sum_{i>1} x_i^2$. Then $\delta^2/6 \leq z \leq \delta^2$ and $\sum_{i>1} x_i^3 \leq z^{3/2}$. Concavity of $\alpha z - \beta z^{3/2}$ therefore gives

$$\sum_{i>1} q(x_i) \geq \min \left\{ \frac{\alpha \delta^2}{6} - \frac{\beta \delta^3}{6\sqrt{6}}, q(\delta) \right\}. \quad (18)$$

Both resulting numerators, after adding $C + q(m)$, decrease as m goes from $1/2$ to $59/100$. For the first one use $q'(m) < 0$ and $2\alpha - 3\beta\delta/\sqrt{6} > 0$ on $0 \leq \delta \leq 1/2$. For the second use $q(m) + q(1-m) = -38/9 + (52/3)m(1-m)$. The positive weight $5 + \sqrt{m} + \sqrt{\delta}$ also decreases. At $m = 59/100$, the two products, divided by D and decreased by seven, are respectively greater than $222/100000$ and $1433/100000$. Together with (11) this proves the seven-dimensional bound.

5.2 Large largest coordinate

Assume $59/100 \leq m < 1$, and put $r = \sqrt{m}$, $\delta = 1 - r^2$, $d = n - 1 \geq 6$. Let $Z = \sum_{i>1} u_i U_i$ have even nonincreasing density g . Its variance is $\delta/12$, so

$$\mathbb{E}Z^2 \geq 2 \int_0^{r/2} t^2 g(r/2) dt \implies C_1 \leq B := \frac{\delta}{r^3}.$$

Moreover $\mathbb{E}Z^4 = \delta^2/48 - \sum_{i>1} u_i^4/120 \leq \delta^2/48$. The graph representation and Markov's inequality give

$$A_n = \frac{1}{r} \mathbb{P}(|Z| \leq r/2) \geq H := \frac{1}{r} \left(1 - \frac{\delta^2}{3r^4} \right). \quad (19)$$

Apply Chebyshev only to indices $i > 1$, and set $W = \sum_{i>1} \sqrt{1 - u_i^2}$. Equation (10) yields

$$\frac{P_n}{2} \geq A_n W - C_1(W/d - \sqrt{\delta}).$$

This expression increases with W , because $C_1 \leq A_n$. Concentrating the tail gives $W \geq W_0 = d - 1 + r$, and $W_0/d > \sqrt{\delta}$ since $r > \sqrt{\delta}$. Consequently

$$\frac{P_n}{2} \geq L_d(r) := H(d - 1 + r) - B \left(\frac{d - 1 + r}{d} - \sqrt{\delta} \right). \quad (20)$$

We prove $L_d(r) > d$. Set $K(r) = 3r^4 + r^3 + r^2 - r - 1$. Then

$$H - 1 = \frac{(1 - r)K(r)}{3r^5}, \quad \frac{H - 1}{B(1 - r)} = \frac{K(r)}{3r^2(1 - r^2)} \geq \frac{47}{192}.$$

Indeed $r > 3/4$, K increases there, $K(3/4) = 47/256$ and $r^2(1 - r^2) \leq 1/4$. It follows that

$$(L_{d+1} - (d+1)) - (L_d - d) = H - 1 - \frac{B(1-r)}{d(d+1)} > 0 \quad (d \geq 6),$$

since $47/192 > 1/42$. It remains to check $d = 6$. With $Q(r) = 31r^4 + 6r^3 + 3r^2 - 12r - 10$, exact algebra gives

$$6r^5(L_6 - 6) = (1 - r) \left[Q(r) + 6r^2(1 + r)\sqrt{1 - r^2} \right]. \quad (21)$$

The polynomial Q increases on $[3/4, 1]$ and $Q(5/6) = 655/1296 > 0$. Thus the bracket is positive for $r \geq 5/6$. On $[\sqrt{59/100}, 5/6]$, its second summand also increases: its derivative has the sign of $2 + 3r - 3r^2 - 4r^3$, which decreases on this interval and equals $11/108 > 0$ at $5/6$. The bracket at $\sqrt{59/100}$ is greater than $706/10000$. This proves (21) and the desired strict inequality for every $n \geq 7$ with positive coordinates and $m < 1$.

5.3 Exact constants and boundary directions

All the finitely many radical signs above have rational certificates. For a positive rational x , set $L_x = \lfloor 10^{24}\sqrt{x} \rfloor / 10^{24}$ and $U_x = L_x + 10^{-24}$. The integer square-root calculation checks $L_x^2 \leq x < U_x^2$. Substituting lower or upper endpoints with the appropriate signs proves the following conservative margins:

Comparison	Strict lower margin
Equation (15)	0.0249
Equation (17)	0.0717
Seven-dimensional spread-tail branch	0.00222
Seven-dimensional concentrated-tail branch	0.01433
Bracket in (21) at $\sqrt{59/100}$	0.0706

The exact rational values and symbolic identities are included in the reproducibility files. These are finite arithmetic certificates for the constants, not grids approximating a continuum inequality.

By Lemma 2.1, the bounds extend to normals with zero coordinates. Their strict lower estimates depend continuously on $m < 1$, so equality cannot appear at such a boundary direction. At $m = 1$ the section is a coordinate cube and has perimeter $2(n-1)$. This completes the theorem for every dimension at least seven.

6 Exact finite certificates in dimensions five and six

By sign, scale and permutation symmetry, it remains to treat $(1, b_1, \dots, b_d)$ with $1 \geq b_1 \geq \dots \geq b_d \geq 0$ and $d = n - 1$. At depth k , put $M = 2^k$ and consider cells

$$l_i/M \leq b_i \leq (l_i + 1)/M, \quad 0 \leq l_d \leq \dots \leq l_1 < M.$$

Bounds hold on the whole box; coverage uses its ordered part. The center has integer-scaled normal $v = (2M, 2l_1 + 1, \dots, 2l_d + 1)$. By Lemma 2.1, its perimeter divided by the following factor is a lower bound on the cell:

$$F_5(M) = \frac{(M+1)(M+2)^3}{M^4}, \quad F_6(M) = \frac{(8M+9)(2M+5)^4}{128M^5}. \quad (22)$$

For $n = 5$ use $\|b - c\|_1 \leq 2/M$ and $\|b - c\|_2 \leq 1/M$; for $n = 6$ use $\|b - c\|_1 \leq 5/(2M)$ and $\|b - c\|_2 \leq \sqrt{5}/(2M) < 9/(8M)$.

6.1 Two ways to certify a cell

The first test is analytic. If

$$\sum_{i=1}^{d-1} l_i - l_d + (d-1) \leq M, \quad (23)$$

then every point satisfies $\sum_i b_i - 2b_d \leq 1$ and lies either in the dominant cone or in the first clipping chamber of Lemma 3.1. If $b_d = 0$, the same condition forces the dominant cone, so no positive-coordinate hypothesis is misapplied. These cells have equality only at the coordinate direction.

The other test uses an exact center calculation. Write $T = \sum_i v_i$, $N^2 = \sum_i v_i^2$, and

$$g_i = \sum_{S \subseteq [n] \setminus \{i\}} (-1)^{|S|} \left(T - 2 \sum_{j \in S} v_j \right)_+^{n-2}.$$

Inclusion-exclusion for the convolution of uniform densities gives

$$P_n(v) = \frac{\sum_i v_i g_i \sqrt{N^2 - v_i^2}}{2^{n-3}(n-2)! \prod_i v_i}. \quad (24)$$

Indeed, after translating the $n-1$ remaining centered uniforms to positive intervals, their density is evaluated at $T/2$. Each g_i is a nonnegative multiple of that density. Replace each root in (24) by $\lfloor 10^6 \sqrt{N^2 - v_i^2} \rfloor / 10^6$. The resulting lower bound P_{low} is compared, by exact integer multiplication, with $2(n-1)F_n(M)$. A cell passes only if

$$P_{\text{low}} > 2(n-1)F_n(M). \quad (25)$$

Thus its entire closed box satisfies the strict perimeter inequality. Integer square-root inequalities verify every floor; floating-point seeds in the generator do not determine the comparison.

6.2 Coverage and verification

Starting with the depth-zero cell, an unaccepted cell is split into its 2^d dyadic children; children with non-descending indices are omitted. Every ordered point is covered: its coordinatewise floor indices at the new depth, with the endpoint one clamped to the last interval, are descending and have the prescribed parent. Closed faces may overlap, which does not affect the lower bound. The complete trees terminate with the following counts and no unresolved leaf:

n	Visited nodes	Analytic leaves	Exact center leaves	Depth
5	756600	6213	698775	6
6	21993217	328808	20884943	7

The five-dimensional CSV lists every leaf. A separate Python program using unbounded integers rechecks every leaf inequality and recursively verifies that the list covers every ordered child. The six-dimensional binary preorder tree uses one byte per visited node. Its separate C++ verifier uses arbitrary-precision integers for large comparisons, integer bisection for square roots, and an iteratively constructed signed-subset list instead of the generator's mask loop. It rejects an unresolved or missing node, excess depth, and trailing nodes. All comparisons are strict integer comparisons; assertions must remain enabled when compiling.

The six-dimensional depth bound is seven, so all coordinates are at most 256 and $T \leq 1536$. Every partial signed density sum is bounded in absolute value by $32 \cdot 1536^4 < 2^{63}$. Each square-root argument is at most $5 \cdot 256^2 \cdot 10^{12} < 10^{18}$, within its exact integer bisection range. Other large verifier quantities use arbitrary precision. The generator’s separate fixed-width bounds are checked as well, but the arbitrary-precision verifier is sufficient to validate the supplied tree.

The archive includes both certificates, all source programs, exact verification outputs, SHA-256 manifests and reproduction instructions. The five-dimensional certificate has 704988 verified leaves and the six-dimensional certificate has 21213751 verified leaves. Supplemental comparisons with a separate rational-density implementation and geometric spot checks are recorded separately; they are not substituted for coverage.

Together, (23) and (25) prove Theorem 1.1 in dimensions five and six, including its equality case. The preceding sections cover all other dimensions. This completes the proof of the stated minimum theorem.

7 Reproducibility and limitations

The supplied source archive identifies the exact certificates and checker programs by SHA-256. The five-dimensional checker uses Python integer arithmetic; the six-dimensional checker requires a C++17 compiler and Boost.Multiprecision. No random sample is used to decide coverage. The negative-moment constants use rational arithmetic and integer square roots; the continuum inequalities are proved in the text. Build the checkers without NDEBUB, since their assertions are part of the verification.

This manuscript was developed with AI assistance in literature search, mathematical derivation, symbolic computation and drafting. The recorded audit belongs to that originating research workflow. Independent peer review and proof-assistant verification are not claimed. Publication of this preprint does not certify novelty or resolve the unrelated parts of the bundled AIM question.

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