

Subgaussian Convex Bodies Need Not Have Subgaussian Projections

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Abstract

Every centrally symmetric convex body is an orthogonal projection of a centrally symmetric convex body whose linear functionals have ψ_2/L_2 ratio at most $2\sqrt{2}$, independently of dimension. The lift is the intersection of a Euclidean ball with a cylinder in a sufficiently large dimension. In particular, for every $n \geq 2$ we give an explicit body $C_n \subset \mathbb{R}^{64n^2}$ with this bound whose n -dimensional coordinate projection is $\frac{1}{2}B_1^n$. The latter has ψ_2/L_2 ratio at least $\sqrt{n+2}/(2\sqrt{2}\log 2)$. Thus dimension-free subgaussianity of uniform volume does not survive geometric projection. This gives a negative answer to the projection subquestion in Paouris's Problem 7 in the AIM list on algorithmic convex geometry. It does not address the zonoid subquestion.

1 The projection question

Let $K \subset \mathbb{R}^d$ be a convex body with barycenter zero, and let μ_K be normalized Lebesgue measure on K . For a random variable Y put

$$\|Y\|_{\psi_2} = \inf\{s > 0 : \mathbb{E} \exp(Y^2/s^2) \leq 2\}.$$

Following the convention in [2, §2.2.2], define

$$b_2(K) = \sup_{\theta \neq 0} \frac{\|\langle \cdot, \theta \rangle\|_{L_{\psi_2}(\mu_K)}}{\|\langle \cdot, \theta \rangle\|_{L_2(\mu_K)}}. \quad (1)$$

The assertion that a family consists of ψ_2 -bodies means that this quantity is bounded by a common constant, independent of dimension. For a single fixed convex body the quantity is always finite. An invertible linear change of variables leaves b_2 unchanged.

Paouris's Problem 7 in [1] asks for classes of ψ_2 -bodies, asks about zonoids, and asks whether projections of ψ_2 -bodies remain ψ_2 -bodies. We treat the last question. We always equip the projected *body* with its own normalized Lebesgue measure, not with the pushforward of μ_K . The pushforward version preserves (1) immediately and is a different assertion.

Write B_p^d for the unit ball of ℓ_p^d and P_E for orthogonal projection onto a linear subspace E .

Theorem 1. *For $n \geq 2$ let $N = 64n^2$ and*

$$C_n = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{N-n} : |x|_2^2 + |y|_2^2 \leq 1, |x|_1 \leq \tfrac{1}{2}\}. \quad (2)$$

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Then C_n is a centrally symmetric convex body,

$$b_2(C_n) \leq 2\sqrt{2}, \quad P_{\mathbb{R}^n} C_n = \frac{1}{2} B_1^n, \quad b_2(P_{\mathbb{R}^n} C_n) \geq \frac{\sqrt{n+2}}{2\sqrt{2\log 2}}.$$

In particular, geometric projections of bodies with a uniformly bounded b_2 need not have a uniformly bounded b_2 .

The failure of a dimension-free ψ_2 bound for B_1^n is known: [2, Proposition 11] determines its directional constants up to absolute factors. We include the elementary one-coordinate estimate needed here. The lift and the distinction between the two measures are the focus of the argument.

2 Conditioning uniform volume on a ball

Lemma 2. *Let U be uniform on B_2^N and let $A \subset B_2^N$ be a measurable centrally symmetric set with $\mathbb{P}(U \in A) \geq 15/16$. If V has the law of U conditioned on $U \in A$, then, for every unit vector θ ,*

$$\mathbb{E}\langle V, \theta \rangle^2 \geq \frac{1}{2(N+2)}, \quad \|\langle V, \theta \rangle\|_{\psi_2} \leq \frac{2}{\sqrt{N+2}}.$$

In particular all its directional ψ_2/L_2 ratios are at most $2\sqrt{2}$.

Proof. Put $Y = \langle U, \theta \rangle$. Rotational symmetry and slicing the ball give the density proportional to $(1-t^2)^{(N-1)/2}$ on $[-1, 1]$. If

$$I_k = \int_{-1}^1 t^{2k} (1-t^2)^{(N-1)/2} dt,$$

integration of the derivative of $t^{2k-1}(1-t^2)^{(N+1)/2}$ gives $I_k/I_{k-1} = (2k-1)/(N+2k)$ for $k \geq 1$. Hence

$$\mathbb{E}Y^{2k} = \frac{(2k-1)!!}{(N+2)(N+4)\cdots(N+2k)} \leq \frac{(2k-1)!!}{(N+2)^k}. \quad (3)$$

Here the empty product, corresponding to $k=0$, is one.

Let $\delta = \mathbb{P}(U \notin A) \leq 1/16$. Cauchy–Schwarz and (3) yield

$$\begin{aligned} \mathbb{E}[Y^2 \mathbf{1}_{\{U \in A\}}] &\geq \frac{1}{N+2} - \sqrt{\frac{3\delta}{(N+2)(N+4)}} \\ &\geq \frac{1-\sqrt{3\delta}}{N+2} \geq \frac{1-\sqrt{3}/4}{N+2} > \frac{1}{2(N+2)}. \end{aligned}$$

Division by $\mathbb{P}(U \in A) \leq 1$ proves the lower second-moment bound. The conditional distribution is centered by symmetry.

By monotone convergence and (3),

$$\mathbb{E} \exp\left(\frac{N+2}{4} Y^2\right) \leq \sum_{k=0}^{\infty} \frac{(2k-1)!!}{4^k k!} = \sum_{k=0}^{\infty} \binom{2k}{k} 8^{-k} = \sqrt{2}.$$

Consequently the conditional expectation of the same exponential is at most $16\sqrt{2}/15 < 2$. This proves the ψ_2 bound and then the stated ratio bound. $\square$

3 Lifting an arbitrary symmetric convex body

Theorem 3. *Suppose $K = -K \subset \mathbb{R}^n$ is a convex body and $rB_2^n \subset K \subset RB_2^n$, where $0 < r \leq R$. For every integer*

$$N \geq 64n(R/r)^2$$

there is a centrally symmetric convex body $L \subset \mathbb{R}^N$ such that

$$P_{\mathbb{R}^n} L = K, \quad b_2(L) \leq 2\sqrt{2}.$$

Proof. Set $K' = K/(2R)$, $\rho = r/(2R)$, and

$$C = B_2^N \cap (K' \times \mathbb{R}^{N-n}).$$

This is a convex body: it is compact, convex, centrally symmetric, and contains a neighborhood of zero. Since $K' \subset \frac{1}{2}B_2^n$, every $x \in K'$ has $(x, 0) \in C$. Thus $P_{\mathbb{R}^n} C = K'$.

For U uniform on B_2^N , put $X = P_{\mathbb{R}^n} U$. By (3), $\mathbb{E}|X|_2^2 = n/(N+2)$. Since $\rho B_2^n \subset K'$, Markov's inequality gives

$$\mathbb{P}(U \notin C) \leq \mathbb{P}(|X|_2 > \rho) \leq \frac{n}{(N+2)\rho^2} = \frac{4n(R/r)^2}{N+2} < \frac{1}{16}.$$

Conditioning U on C gives exactly normalized volume on C . Lemma 2 proves $b_2(C) \leq 2\sqrt{2}$. Now take $L = 2RC$ and use dilation invariance. $\square$

4 The explicit bad projections

We prove Theorem 1. Convexity, central symmetry and nonempty interior follow at once from (2). The projection is $\frac{1}{2}B_1^n$ because $|x|_2 \leq |x|_1$. For completeness, the removed-volume estimate in this example is particularly direct. If U is uniform on B_2^N and X is its first n coordinates, then

$$\mathbb{P}(|X|_1 > \tfrac{1}{2}) \leq 4\mathbb{E}|X|_1^2 \leq 4n\mathbb{E}|X|_2^2 = \frac{4n^2}{N+2} < \frac{1}{16}.$$

Lemma 2 gives the asserted bound for C_n .

Let Z be uniform on B_1^n . Its first coordinate has density $\frac{n}{2}(1-|t|)^{n-1}$ on $[-1, 1]$, obtained by taking $(n-1)$ -dimensional sections. Therefore

$$\mathbb{P}(|Z_1| \geq t) = (1-t)^n \quad (0 \leq t \leq 1), \quad \mathbb{E}Z_1^2 = 2 \int_0^1 t(1-t)^n dt = \frac{2}{(n+1)(n+2)}.$$

For every $s > 0$ satisfying $\mathbb{E} \exp(Z_1^2/s^2) \leq 2$, the event $|Z_1| \geq 1/2$ gives

$$2 \geq 2^{-n} \exp(1/(4s^2)).$$

Taking logarithms and then the infimum over admissible s yields

$$\|Z_1\|_{\psi_2} \geq \frac{1}{2\sqrt{(n+1)\log 2}}, \quad \frac{\|Z_1\|_{\psi_2}}{\|Z_1\|_2} \geq \frac{\sqrt{n+2}}{2\sqrt{2\log 2}}.$$

Dilation invariance proves the last assertion of Theorem 1.

5 Normalization and scope

The counterexample also admits isotropic normalization. Sign changes and permutations in the first coordinate block, together with orthogonal transformations in the second, force the covariance of uniform volume on C_n to be $\text{diag}(aI_n, bI_{N-n})$ for some $a, b > 0$. The linear map

$$\text{diag}(a^{-1/2}I_n, b^{-1/2}I_{N-n})$$

makes the covariance the identity and sends the first-block projection to a scalar multiple of B_1^n . Both b_2 estimates are unchanged. A further common dilation gives volume one, if that isotropic convention is preferred.

For the unnormalized body C_n , the pushforward of μ_{C_n} onto the first block has density proportional to

$$\mathbf{1}_{\{|x|_1 \leq 1/2\}}(1 - |x|_2^2)^{(N-n)/2}.$$

This strongly varying fiber-volume weight is absent from normalized volume on the projection. The pushforward retains the bound $2\sqrt{2}$ for linear functionals; the uniform law on its support does not.

There is no conflict with the estimates for uniformly 2-convex bodies in Klartag–Milman [3], Lemma 3.4 and Proposition 3.5. The bodies (2) have nontrivial boundary segments and are not uniformly convex. Nor is projection of the auxiliary subgaussian body $\Psi_2(K)$, studied for example in [4, Theorem 1.3], the same operation as computing ψ_2 norms under uniform volume on $P_E K$.

No zonoid representation of C_n is asserted. The result does not settle whether all zonoids have dimension-free b_2 , or classify all ψ_2 -bodies. The dimension bound in Theorem 3 is not claimed optimal.

This unrefereed manuscript was prepared with AI assistance in the derivation, source comparison, writing and exact-arithmetic checks. The written argument is the proof; finite regression checks are not a proof-assistant verification. No independent review or absolute priority claim is made.

References

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