

A Geometric-Mean Proof of the Spherical Simplex Volume-Product Bound

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Abstract

The product of the normalized solid angles of a simplicial cone in $\mathbb{R}^n$ and its positive dual is at most 4^{-n} , with equality exactly for orthogonal images of the positive orthant. We give a short proof by applying the classical geometric-mean Prékopa–Leindler inequality to two inverse positive definite quadratic forms. This answers the upper-bound question in Problem 23 of the 2010 AIM workshop list on Mahler’s conjecture and duality. The functional inequality and its equality criterion are established ingredients, not new results of this note. We also compute the second variation in unit-diagonal Gram coordinates, obtaining an explicit positive quadratic deficit and a dimension-dependent local stability estimate. No absolute priority claim is made for the spherical corollary or the local expansion.

1 The spherical question

Write $\sigma_{n-1} = \text{vol}_{n-1}(S^{n-1})$. For a closed cone $C \subset \mathbb{R}^n$ define its positive dual and its normalized solid angle by

$$C^* = \{y : \langle x, y \rangle \geq 0 \text{ for every } x \in C\}, \quad \omega(C) = \frac{\text{vol}_{n-1}(C \cap S^{n-1})}{\sigma_{n-1}}.$$

A full-dimensional simplicial cone has the form $C = A\mathbb{R}_+^n$, where A is invertible and $\mathbb{R}_+^n = [0, \infty)^n$. Its spherical section is a nondegenerate spherical simplex.

Problem 23, attributed to H. Koenig in the AIM workshop list [1], asks whether the spherical volume product of such a simplex and its positive spherical dual is at most the corresponding product for the orthant. The list states the result in dimension three. The frozen UnsolvedMath v1.6.0 record AIM-CONVEX_GEOMETRY-0019 [3] contains OCR errors: the item number appears as 3, the upper-bound sign is damaged, and the nonnegative dual condition is printed as strictly positive. We use the primary PDF’s formulation. Changing the boundary convention would not change these solid angles.

Theorem 1. *For every full-dimensional simplicial cone $C \subset \mathbb{R}^n$,*

$$\omega(C)\omega(C^*) \leq 4^{-n}. \tag{1}$$

Equality holds if and only if C is an orthogonal image of $\mathbb{R}_+^n$. Equivalently, for $Q = C \cap S^{n-1}$ and its positive spherical dual $Q^\circ = C^* \cap S^{n-1}$,

$$\text{vol}_{n-1}(Q) \text{vol}_{n-1}(Q^\circ) \leq 4^{-n} \sigma_{n-1}^2.$$

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The proof below is a direct Gaussian specialization of the geometric-mean integration inequality stated, including its equality case, by Fradelizi and Meyer [2, Proposition 1]. They attribute this classical form of Prékopa–Leindler to Borell, Ball and Uhrin. We reproduce the logarithmic change of variables to make the application transparent. We do not claim a new functional inequality or that this spherical corollary was previously unknown.

2 Geometric-mean integration and inverse quadratic forms

Lemma 2 (Classical geometric-mean inequality). *Let f, g, h be nonnegative integrable functions on $(0, \infty)^n$. If, for every $x, y \in (0, \infty)^n$,*

$$f(x)g(y) \leq h(\sqrt{x_1 y_1}, \dots, \sqrt{x_n y_n})^2, \quad (2)$$

then

$$\left(\int f\right)\left(\int g\right) \leq \left(\int h\right)^2. \quad (3)$$

For $h(x) = e^{-|x|^2/2}$ and continuous positive f, g , equality implies

$$f(x) = de^{-|Dx|^2/2}, \quad g(x) = d^{-1}e^{-|D^{-1}x|^2/2}, \quad (4)$$

where D is positive diagonal and $d > 0$.

Proof. For $u \in \mathbb{R}^n$ set

$$F(u) = f(e^{u_1}, \dots, e^{u_n})e^{\sum_i u_i},$$

and define G, H similarly. The exponential substitution gives $\int_{\mathbb{R}^n} F = \int_{(0, \infty)^n} f$, and likewise for g, h . Equation (2) becomes $F(u)G(v) \leq H((u+v)/2)^2$. The usual Prékopa–Leindler inequality at weight 1/2 proves (3).

For the equality conclusion, extend f, g, h by reflection in each coordinate to unconditional functions on $\mathbb{R}^n$; their integrals acquire the same factor 2^n . Proposition 1 of [2] says that equality forces reciprocal coordinate dilations and reciprocal scalar multipliers of h . Its geometric-mean concavity condition on h holds because $|x|^2 + |y|^2 \geq 2 \sum_i x_i y_i$. This gives (4) almost everywhere, hence everywhere on the positive orthant by continuity. $\square$

No log-concavity or coordinatewise monotonicity of f or g is required. In particular, the next application permits either sign in the off-diagonal entries of the quadratic forms.

Proposition 3. *For a real symmetric positive definite $n \times n$ matrix M , put*

$$J(M) = \int_{(0, \infty)^n} e^{-x^T M x / 2} dx.$$

Then

$$J(M)J(M^{-1}) \leq (\pi/2)^n, \quad (5)$$

with equality if and only if M is diagonal.

Proof. Both integrals are finite by positive definiteness. For all x, y ,

$$x^T M x + y^T M^{-1} y - 2x^T y = (Mx - y)^T M^{-1} (Mx - y) \geq 0. \quad (6)$$

Thus $f(x) = e^{-x^T M x/2}$, $g(y) = e^{-y^T M^{-1} y/2}$ and $h(z) = e^{-|z|^2/2}$ satisfy (2). Lemma 2 and $\int_0^\infty e^{-t^2/2} dt = \sqrt{\pi/2}$ prove (5).

If equality holds, (4) gives $x^T M x = |Dx|^2 - 2 \log d$ on the positive orthant. Letting $x \rightarrow 0$ gives $d = 1$. Equality of the two quadratic polynomials on an open set then gives $M = D^2$, so M is diagonal. Conversely, if M is positive diagonal, separation of the one-dimensional integrals gives

$$J(M) = \frac{(\pi/2)^{n/2}}{\sqrt{\det M}}, \quad J(M^{-1}) = (\pi/2)^{n/2} \sqrt{\det M},$$

which proves equality. $\square$

3 Application to spherical simplices

Proof of Theorem 1. Write $C = A\mathbb{R}_+^n$ and $M = A^T A$. Directly from the definition,

$$C^* = \{z : A^T z \in \mathbb{R}_+^n\} = A^{-T} \mathbb{R}_+^n. \quad (7)$$

A standard Gaussian vector has uniform spherical direction. Changing variables $z = Ax$ in its density therefore gives

$$\omega(C) = (2\pi)^{-n/2} |\det A| J(M).$$

Applying the same observation to A^{-T} gives

$$\omega(C^*) = (2\pi)^{-n/2} |\det A|^{-1} J(M^{-1}),$$

because $A^{-1} A^{-T} = (A^T A)^{-1}$. The determinants cancel, and Proposition 3 yields

$$\omega(C)\omega(C^*) = (2\pi)^{-n} J(M)J(M^{-1}) \leq (2\pi)^{-n} (\pi/2)^n = 4^{-n}.$$

Equality means that $A^T A$ is diagonal, or that the generating columns of A are pairwise orthogonal. Dividing them by their positive lengths does not change the cone and produces an orthogonal matrix. The converse follows because an orthant and its positive dual each have solid angle 2^{-n} . Finally, multiplying by σ_{n-1}^2 gives the spherical statement. $\square$

Remark 4. The argument also covers the natural degenerate interpretation of the source's hyperplane description. A chamber cut out by n independent homogeneous linear inequalities is an invertible image of the orthant. If those normals are dependent, their common kernel contains a line in the chamber. The positive dual is contained in the orthogonal complement of that line, so its ambient spherical measure is zero. A lower-dimensional cone itself has zero ambient spherical measure. The bound here concerns simplicial cones, not arbitrary nonsimplicial cones.

4 The local quadratic deficit

For positive definite M define

$$p(M) = (2\pi)^{-n} J(M)J(M^{-1}).$$

Positive diagonal congruence preserves p : substituting $x = D^{-1}u$ gives $J(DMD) = (\det D)^{-1} J(M)$, while $J((DMD)^{-1}) = (\det D) J(M^{-1})$. Thus we may normalize the diagonal of a Gram matrix to one without changing the cone product.

Theorem 5. Let $B = (b_{ij})$ be real symmetric with zero diagonal, and set

$$m = \frac{2}{\pi}, \quad S_2 = \sum_{i < j} b_{ij}^2, \quad T = \sum_i \left(\sum_j b_{ij} \right)^2.$$

For sufficiently small real t ,

$$\log(4^n p(I + tB)) = -[m(1 - m)S_2 + m(m - \frac{1}{2})T]t^2 + O_B(t^3). \quad (8)$$

For each fixed n there exists $\delta_n > 0$ such that every symmetric zero-diagonal E with $\|E\|_F < \delta_n$ satisfies

$$\log(4^n p(I + E)) \leq -\frac{m(1 - m)}{4} \|E\|_F^2. \quad (9)$$

Proof. Let $X_1, \dots, X_n$ be independent half-normal variables, so that $\mathbb{E}X_i = \sqrt{m}$ and $\mathbb{E}X_i^2 = 1$. Put $q = X^T B X$, $r = X^T B^2 X$. Integrating against their joint density gives

$$\frac{J(I + tB)}{J(I)} = \mathbb{E}e^{-tq/2}.$$

Since $(I + tB)^{-1} = I - tB + t^2 B^2 + O_B(t^3)$, expansion of the two logarithms yields

$$\begin{aligned} \log \frac{J(I + tB)}{J(I)} &= -\frac{t}{2} \mathbb{E}q + \frac{t^2}{8} \text{Var}(q) + O_B(t^3), \\ \log \frac{J((I + tB)^{-1})}{J(I)} &= \frac{t}{2} \mathbb{E}q - \frac{t^2}{2} \mathbb{E}r + \frac{t^2}{8} \text{Var}(q) + O_B(t^3). \end{aligned}$$

These expansions are legitimate under the integral: for small $|t|$, both quadratic-form matrices have eigenvalues at least $1/2$, and their derivatives through order three are dominated by a polynomial times $e^{-|x|^2/4}$.

Independence gives

$$\mathbb{E}r = 2(1 - m)S_2 + mT. \quad (10)$$

To compute the other moment, write $q/2 = \sum_{i < j} b_{ij} X_i X_j$. A single edge variable $X_i X_j$ has variance $1 - m^2$. Two distinct edges have covariance zero when disjoint and $m(1 - m)$ when incident. The sum of coefficient products over unordered incident edge pairs is $(T - 2S_2)/2$. Consequently

$$\frac{1}{4} \text{Var}(q) = (1 - m)^2 S_2 + m(1 - m)T. \quad (11)$$

The quadratic coefficient of the sum of the logarithms is $-\mathbb{E}r/2 + \text{Var}(q)/4$. Substitution of (10)–(11) proves (8).

Both coefficients in its negative quadratic form are positive because $1/2 < m < 1$. Also $S_2 = \|B\|_F^2/2$. At fixed dimension the third derivative bound can be chosen uniformly on the compact unit sphere of symmetric zero-diagonal matrices. For $E = tB$, $\|B\|_F = 1$, take the resulting remainder to be at most $m(1 - m)t^2/4$ by decreasing the permitted $|t|$. This proves (9). The zero perturbation is immediate. $\square$

For $n = 2$, the coefficient in (8) is $-4b_{12}^2/\pi^2$, in agreement with the exact wedge formula. Indeed, if a planar wedge has opening $\alpha \in (0, \pi)$, its positive dual has opening $\pi - \alpha$, and

$$\omega(C)\omega(C^*) = \frac{\alpha(\pi - \alpha)}{4\pi^2}.$$

Orthogonal products multiply solid angles. These elementary wedge-product cases are already present in the dataset's research aid; they are not the contribution claimed here. No explicit dimension-independent stability radius is asserted.

5 Scope and reproducibility

Theorem 1 answers the full upper-bound question in the primary AIM item. Its proof is an application of a classical theorem; it does not establish historical priority for the corollary. Theorem 5 provides an additional explicit calculation, not an assumption in the global proof. Neither theorem is a claim about the general Euclidean Mahler conjectures.

The accompanying script checks 104 exact symbolic or rational identities and 46 numerical sanity conditions, with identical results in ordinary and optimized Python. The exact checks include the inverse Gram identity, determinant cancellation and half-normal moment polynomials through dimension six. The numerical tests use approximate Gaussian distribution calculations and are not a proof or certified integration. The all-dimensional argument is the written proof above.

AI assistance was used for derivation, literature search, computation and drafting. The author is responsible for the manuscript. This is an unrefereed preprint; no independent review, proof-assistant verification or absolute novelty certification is claimed.

References

- [1] American Institute of Mathematics, *Mahler’s conjecture and duality in convex geometry*, workshop problem list, 9–13 August 2010, Problem 23, p. 4. <https://aimath.org/WWN/mahlerduality/mahlerduality.pdf>.
- [2] M. Fradelizi and M. Meyer, Some functional forms of Blaschke–Santaló inequality, *Math. Z.* **256** (2007), 379–395. doi:10.1007/s00209-006-0078-z. Preprint: [arXiv:math/0609553](https://arxiv.org/abs/math/0609553), Proposition 1.
- [3] ulamai, *UnsolvedMath*, frozen version 1.6.0, record AIM-CONVEX_GEOMETRY-0019, commit `c6e7f41b4eb6` (full hash in the source archive). <https://huggingface.co/datasets/ulamai/UnsolvedMath>.