

Sharp Ambiguity Bounds for Surviving Conditional Odds Ratios

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Abstract

Fix the support of a binary d -way probability table, its feasible interior one-way margins, and every conditional pairwise odds ratio whose four cells remain positive. We show that the largest possible dimension of the resulting family, over all supports, is $\lceil 2^{d+1}/3 \rceil - d - 1$ for $d \geq 2$. The same maximum is attained with uniform margins. For $d \geq 3$, equality occurs precisely on the classical extremal square-free layer sets, up to cube automorphisms. The argument combines classical mixed coordinates with a pivot-deletion lemma and the Kostochka–Johnson–Entringer vertex theorem. We also give a four-variable uniform-margin family of dimension five with one surviving ratio. Two exact implementations verify all nonempty supports through dimension four. This is a conditional-ratio result, not a classification of arbitrary collapsed marginal ratios or a full solution of the broad AIM specification question.

1 Supported tables and the extremal result

Let S be a nonempty subset of $V_d = \{0, 1\}^d$. A table on S has $p_x > 0$ for $x \in S$, $p_x = 0$ elsewhere, and $\sum p_x = 1$. Let A_S be the matrix with column $(1, x_1, \dots, x_d)^T$ at x . For each coordinate square wholly contained in S , let C_S have the row with entries $(1, -1, -1, 1)$ in corner order $(00, 10, 01, 11)$ and zeros elsewhere. Thus $C_S A_S^T = 0$. We call these the *surviving* conditional pairwise odds ratios. Ratios involving an absent cell are not counted as additional equations. In particular this convention does not impose polynomial conditional independence equations on squares with missing corners.

For b with first entry one and $q > 0$, consider

$$\mathcal{F}(S, b, q) = \{p \in \mathbb{R}_{>0}^S : A_S p = b, C_S \log p = \log q\}. \quad (1)$$

An *interior feasible* b means that such a strictly positive table exists before the ratio equations are imposed. A compatible q means $\log q \in \text{im } C_S$. The usual mixed-coordinate theorem implies that the fiber is analytically diffeomorphic to $\mathbb{R}^{\delta(S)}$, where

$$\delta(S) = |S| - \text{rank } A_S - \text{rank } C_S. \quad (2)$$

We recall a proof in Section 3 to make the support and positivity assumptions explicit.

Theorem 1.1. *For $d \geq 2$, put $E_d = \lceil 2^{d+1}/3 \rceil$. Then*

$$\max_{\emptyset \neq S \subseteq V_d} \delta(S) = E_d - d - 1.$$

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The same maximum holds if S is restricted to supports admitting a strictly positive table with uniform one-way margins. For $d \geq 3$, equality holds precisely when, up to coordinate permutations and complementations,

$$S = \{x \in V_d : |x| \not\equiv j \pmod{3}\}, \quad |S| = E_d. \quad (3)$$

In particular every equality support has no surviving square. If at least one ratio survives, then $\delta(S) \leq E_d - d - 2$ for $d \geq 3$; this improved bound is attained in dimension four.

Fienberg's Question 6 in the AIM computational algebraic statistics list [1] asks broadly about complete and partial specifications by odds ratios. Its preceding question distinguishes local and collapsed ratios. Our theorem addresses a precise binary conditional-ratio subproblem and does not resolve the general request.

The ingredients are established theory: mixed coordinates on incomplete tables [2, Section 5], and the extremal cube theorem of Kostochka and Johnson–Entringer [4], stated with its equality case in [3, Theorem 1]; see also the short proof in [5]. Fontana, Perrone and Rapallo [6, Sections 3–5] study structural zeros, margin feasibility and failures of uniqueness in three variables. The contribution developed here is the pivot-deletion transfer to the sharp all-dimensional ambiguity bound and its equality case. The cited mixed-coordinate theorem, cube theorem and three-variable failures are not new results of this note. No absolute-priority claim is made.

2 The deletion argument

Lemma 2.1. *For every nonempty $S \subseteq V_d$ there is a square-free subset $T \subseteq S$ such that*

$$\text{rank } A_T = \text{rank } A_S, \quad |T| = |S| - \text{rank } C_S, \quad \delta(S) = |T| - \text{rank } A_T.$$

Moreover, no nonzero vector of row C_S is supported on T .

Proof. Take a reduced row-echelon basis B of row C_S , with pivot coordinates P , and let $T = S \setminus P$. Restriction to P is injective on row C_S . A coordinate square in T would supply a nonzero row of C_S vanishing on P , a contradiction. Each pivot row of B has a single nonzero pivot entry, equal to one. Since $A_S B^T = 0$, it expresses the associated column of A_S as a linear combination of the columns indexed by T . Thus the column span is unchanged by deleting P . The cardinality and deficiency identities follow. $\square$

Equivalently, clearing pivot entries gives a vector-space isomorphism $\ker A_S / \text{row } C_S \simeq \ker A_T$. This need not be an isomorphism of integral lattices and is not an assertion about Markov bases.

Lemma 2.2. *If $T \subseteq V_d$ is nonempty and $\text{rank } A_T = r$, then $|T| \leq 2^{r-1}$.*

Proof. The affine hull has dimension $r - 1$. Choose $r - 1$ coordinate functionals independent on its direction space. Their joint projection is injective on the affine hull and sends T into $\{0, 1\}^{r-1}$. $\square$

The classical vertex theorem states that a coordinate-square-free subset of V_d has at most E_d points. Every equality set is of the form (3) after a cube automorphism [3, Theorem 1]. We use both the bound and this rigidity statement.

Proof of Theorem 1.1. Let $r = \text{rank } A_S$ and choose T as in Lemma 2.1. If $r = d + 1$, the vertex theorem gives $\delta(S) = |T| - d - 1 \leq E_d - d - 1$. If $r \leq d$ and $d \geq 3$, Lemma 2.2 gives

$$\delta(S) \leq 2^{r-1} - r \leq 2^{d-1} - d < E_d - d - 1.$$

Here $2^{r-1} - r$ is nondecreasing for integral $r \geq 1$, and $E_d > 2^{d-1} + 1$. For $d = 2$, both possible rank cases give $\delta(S) \leq 0$, which is the claimed bound.

Choose j minimizing $\sum_{k \equiv j \pmod{3}} \binom{d}{k}$, and take the set (3). The elementary roots-of-unity filter gives

$$|S| = E_d = \begin{cases} (2^{d+1} + 1)/3, & d \text{ even}, \\ (2^{d+1} + 2)/3, & d \text{ odd}. \end{cases}$$

Every square visits three consecutive weight levels, so C_S is empty. For $d \geq 3$, the inequality $|S| > 2^{d-1}$ and Lemma 2.2 give $\text{rank } A_S = d + 1$. This attains the bound.

To obtain uniform margins, let K be the allowed weights. For $d \geq 3$, $\min K \leq 1 < d/2 < d - 1 \leq \max K$. Choose strictly positive λ_k , $k \in K$, summing to one with $\sum k\lambda_k = d/2$. One may put small positive masses on all intermediate levels and then adjust the two extreme masses. Put $p_x = \lambda_{|x|} / \binom{d}{|x|}$. Permutation symmetry and the expected total weight give every coordinate mean $1/2$. Complementing coordinates preserves these margins. For $d = 2$, the full uniform square already attains the maximum zero.

It remains to exclude equality supports with squares when $d \geq 3$. Equality forces $r = d + 1$ and $|T| = E_d$. By the classical rigidity statement we can assume T has the form (3). If $v \in S \setminus T$ has weight w , then $w \equiv j \pmod{3}$. Flipping two zero coordinates or two one coordinates of v produces a square whose other three vertices lie in T . There are

$$\binom{w}{2} + \binom{d-w}{2}$$

such squares. This is at least two for $d \geq 4$. For $d = 3$ the only extremal residue is $j = 0$, so $w = 0$ or 3 and the number is three. The contrast rows of two distinct such squares have coefficient one at v and no other entries outside T . Their difference is nonzero and supported on T , contradicting Lemma 2.1. Thus $S = T$. The converse was already proved. The nonvacuous example in Section 4 completes the last assertion. $\square$

3 Classical mixed coordinates on a fixed support

Proposition 3.1. *Let A be a full-row-rank real matrix on m cells, with the constant vector in its row space. Suppose $P_b = \{p \geq 0 : Ap = b\}$ is a probability polytope containing a strictly positive point. If $CA^T = 0$, then*

$$\{p > 0 : Ap = b, C \log p = \log q\}$$

is nonempty exactly when $\log q \in \text{im } C$. If nonempty, it is analytically diffeomorphic to $\mathbb{R}^{m - \text{rank } A - \text{rank } C}$. For integer A, C , a rational basis of the missing contrast quotient, with denominators cleared, supplies exactly this many additional homogeneous cell ratios to determine a unique table.

Proof. Choose independent rows D of C and extend them to a row basis $B = (D; E)$ of $\ker A$. Given arbitrary $\ell \in \mathbb{R}^{m - \text{rank } A}$, choose z with $Bz = \ell$ and minimize

$$F_z(p) = \sum_{i=1}^m p_i (\log p_i - z_i)$$

over P_b , using $0 \log 0 = 0$. Compactness and strict convexity give a unique minimizer. It is strictly positive: moving a boundary minimizer toward a positive feasible point introduces a negative term

of order $t \log(1/t)$ from its zero coordinates, while the remaining terms change by $O(t)$. Stationarity at this interior minimizer gives $\log p + \mathbf{1} - z \in \text{row } A$, and hence $B \log p = \ell$. Conversely this condition implies stationarity and uniqueness. Thus $p \mapsto B \log p$ is a bijection from the positive part of P_b to $\mathbb{R}^{m-\text{rank } A}$.

Let the columns of U form a basis of $\ker A$. In affine coordinates $p = p_0 + U\theta$, the derivative of $U^T \log p$ is $U^T \text{diag}(1/p_i)U$, positive definite. The map $B \log p$ is an invertible linear transformation of this map, so the analytic inverse function theorem gives a global analytic inverse. Fixing $D \log p$ leaves $E \log p$ arbitrary. The relations among the redundant rows of C are exactly the condition $\log q \in \text{im } C$. Finally, independent added kernel contrasts reduce the remaining dimension one at a time; fewer cannot give a complete specification. Rational bases exist for integer matrices. $\square$

This is the established mixed-parameter phenomenon on possibly incomplete tables; compare [2, Theorems 5.1–5.2]. It is not a new uniqueness theorem. It proves the global fiber statement, which would not follow from a Jacobian rank calculation alone. No assertion is made about solutions on a larger support obtained by clearing denominators at zero cells.

4 Four-variable examples and reproducibility

The extremal support of weights 1, 2, 4 has eleven cells. Give every weight-one and weight-two cell mass $1/12$ and the cell 1111 mass $1/6$. These probabilities sum to one and each coordinate has mean $1/2$. There are no surviving squares and the fixed-margin family has dimension six. Its ratio list is empty.

For a nonempty ratio list, identify a cell with the integer $x_1 + 2x_2 + 4x_3 + 8x_4$ and take

$$S = \{0, 1, 3, 4, 6, 7, 8, 10, 11, 12, 13\}.$$

Its only square is $\{0, 4, 8, 12\}$. The matrices have ranks five and one, respectively. In the indicated cell order, the vector

$$p = \frac{1}{2628} (50, 55, 374, 365, 75, 395, 365, 75, 395, 384, 95)$$

has uniform margins and all entries positive. Its surviving ratio is $p_0 p_{12} / (p_4 p_8) = 768/5329$. Proposition 3.1 also shows that every positive value of this ratio, including one, has a five-dimensional fiber with the same uniform margins. This is not a claim that all conditional independence equations, including equations on incomplete faces, hold.

The accompanying Node program uses primitive integer elimination with checked safe integer operations; a separately written Python program uses exact rational Gauss–Jordan elimination and a different face generator. They agree for every nonempty support in dimensions 2, 3, 4, altogether $15 + 255 + 65535 = 65805$ cases. For $d = 4$ the counts by deficiency are

δ	0	1	2	3	4	5	6
supports	11953	19260	18996	11038	3680	592	16.

All sixteen maximizers are translates of the extremal layer set. The rational checker verifies the deletion lemma for every enumerated support and fifty deterministic test supports in dimensions five and six. The computation supplements the proof and does not establish the arbitrary-dimensional theorem by itself.

The broad AIM question still includes choices beyond those treated here. Collapsed marginal ratios are ratios of sums of cells and need not be linear in cell logarithms. They remain outside the present classification. The literature search did not establish priority for the extremal transfer.

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