# Lacunary Counterexamples to a Distinct-Summand Freiman Container Problem

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**Abstract**

Fix $K > 1$. We construct finite nonempty sets $A, B \subseteq \mathbb{Z}$ with $|A| > |B|$ and $|A + B| < K|A|$ for which $B$ is not contained in any generalized arithmetic progression of bounded rank and size $O_K(|A|)$. More strongly, for every prescribed rank bound $d$ and size constant $C$, one counterexample defeats all progressions of rank at most $d$ and cardinality at most $C|A|$. The construction is $B = \{0, 1, N, \ldots, N^{r-1}\}$ and $A = kB$. Uniqueness of base-$N$ digits gives exact polynomial growth $|sB| = \binom{s+r}{r}$ for $s < N$, whereas every rank-$d$ generalized arithmetic progression has $h$-fold growth at most $h^d$. This answers Problem 2.5 from the 2004 AIM list on recent trends in additive combinatorics in the negative for every $K > 1$.

## 1 Introduction

For finite sets $A, B \subseteq \mathbb{Z}$, Tao's Problem 2.5 in the 2004 AIM list asks whether

$$|A| > |B|, \qquad |A + B| < K|A| \tag{1.1}$$

forces a bounded-rank generalized arithmetic progression $P$ containing $B$, together with a set $X$, such that

$$A \subseteq P + X, \qquad |P + X| \leq c(K)|A|. \tag{1.2}$$

See [1, Problem 2.5]. Croot and Lev repeated the problem under the title "Freiman's theorem for distinct set summands" and recorded positive boundaries when the two summands have equal or comparable size [3, Section 4.5]. The unrestricted asymmetric question has a simple lacunary obstruction.

We use the standard notation

$$sB = \{b_1 + \cdots + b_s : b_i \in B\}.$$

A generalized arithmetic progression (GAP) of rank $q$ is a set

$$P = \{a_0 + x_1 v_1 + \cdots + x_q v_q : 0 \leq x_i < L_i\}. \tag{1.3}$$

No properness is assumed.

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**Theorem 1.1.** *Fix $K > 1$. For every integer $d \geq 0$ and every real $C \geq 1$, there exist finite nonempty sets $A, B \subseteq \mathbb{Z}$ satisfying*

$$|A| > |B|, \qquad |A + B| < K|A|, \tag{1.4}$$

*such that no GAP $P$ of rank at most $d$ can satisfy*

$$B \subseteq P, \qquad |P| \leq C|A|. \tag{1.5}$$

*Consequently there are no such $P$ and $X \subseteq \mathbb{Z}$ satisfying $B \subseteq P$, $A \subseteq P + X$, and $|P + X| \leq C|A|$.*

Taking $d = \lceil c(K) \rceil$ and $C = c(K)$ disproves the proposed uniform conclusion. The examples lie in the highly unbalanced regime, so they do not conflict with the positive comparable-size cases.

## 2 A growth bound for generalized progressions

**Lemma 2.1.** *If $P$ is a GAP of rank $q$, proper or improper, then for every integer $h \geq 1$,*

$$|hP| \leq h^q |P|. \tag{2.1}$$

*Proof.* An element of $hP$ has the form

$$ha_0 + y_1 v_1 + \cdots + y_q v_q, \qquad 0 \leq y_i \leq h(L_i - 1).$$

Write $y_i = x_i + t_i L_i$, where $0 \leq x_i < L_i$ and $0 \leq t_i < h$. For each of the at most $h^q$ vectors $(t_1, \ldots, t_q)$, the corresponding elements lie in one translate of $P$. Hence $hP$ is covered by at most $h^q$ translates of $P$, proving (2.1). Collisions in the coefficient map do not affect the argument. □

## 3 The lacunary construction

*Proof of Theorem 1.1.* Choose an integer $r > d$, and then an integer $k \geq 2$ such that

$$\frac{r}{k+1} < K - 1. \tag{3.1}$$

Put

$$M = \binom{k+r}{r}.$$

Because $r > d$, there is an integer $h \geq 1$ so large that

$$\binom{h+r}{r} > CMh^d. \tag{3.2}$$

Finally choose an integer $N > \max\{k+1, h\}$ and define

$$B = \{0, 1, N, N^2, \ldots, N^{r-1}\}, \qquad A = kB. \tag{3.3}$$

For every integer $s < N$, uniqueness of base-$N$ digits identifies $sB$ with the vectors

$$(c_1, \dots, c_r) \in \mathbb{Z}_{\geq 0}^r, \qquad c_1 + \cdots + c_r \leq s.$$

Stars and bars therefore gives the exact count

$$|sB| = \binom{s+r}{r}. \tag{3.4}$$

In particular,

$$|A| = \binom{k+r}{r} = M, \qquad |A+B| = \binom{k+r+1}{r},$$

and hence

$$\frac{|A+B|}{|A|} = \frac{k+r+1}{k+1} = 1 + \frac{r}{k+1} < K. \tag{3.5}$$

Moreover, $|B| = r+1$, while $k \geq 2$ gives

$$|A| = \binom{k+r}{r} > r+1 = |B|. \tag{3.6}$$

Suppose a GAP $P$ of rank $q \leq d$ contained $B$ and satisfied $|P| \leq C|A| = CM$. Lemma 2.1 would imply

$$|hB| \leq |hP| \leq h^q |P| \leq CMh^d. \tag{3.7}$$

But $h < N$, so (3.4) and (3.2) give

$$|hB| = \binom{h+r}{r} > CMh^d,$$

contradicting (3.7). This proves (1.5).

If also $A \subseteq P+X$, then $X$ is nonempty. For any $x \in X$, the translate $P+x$ lies in $P+X$, so $|P| \leq |P+X|$. Therefore $|P+X| \leq C|A|$ would imply the already impossible small-container condition (1.5). The theorem follows. □

# 4 Deduction for the AIM formulation

Suppose the function $c(K)$ proposed in Problem 2.5 existed. If $c(K) < 1$, then $A \subseteq P+X$ and $|P+X| \leq c(K)|A|$ are already incompatible. Otherwise apply Theorem 1.1 with

$$d = \lceil c(K) \rceil, \qquad C = c(K).$$

The resulting pair satisfies the hypotheses (1.1) but admits no conclusion of the form (1.2). Thus the answer is negative for every $K > 1$.

The proof uses only polynomial sumset growth of a lacunary set and the elementary growth estimate for GAPs. It does not address variants in which the container itself is allowed to have small doubling without being a bounded-rank progression; compare the variants collected in [2].

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## References

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[2] Béla Bajnok. Additive combinatorics: A menu of research problems, 2017.

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