A Finite Gap Bound for Local Progression-Free Density

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Abstract

Fix $s \geq 3$ and consider increasing integer sequences whose every s consecutive terms contain no nontrivial three-term arithmetic progression. We show that every admissible gap sequence can be decreased coordinatewise to an admissible sequence with all gaps at most $B_s = 1 + 2\binom{s}{3}$. Consequently the unrestricted maximum density is the reciprocal of the minimum cycle mean in an explicitly defined finite graph. In particular, it is rational, is computable for each s, and is attained periodically with integer period at most B_s^{s-1}. A nonnegative defect derived from this graph characterizes all extremizers, including those with unbounded gaps, for upper, lower, natural and upper Banach density. A short analytic potential proves the sharp value $4/9$ for $5 \leq s \leq 8$. The general answer is an effective finite characterization, not a short closed formula or a sharp large-s asymptotic.

1 The problem and the gap formulation

Call $A = (a_1 < a_2 < \cdots)$ *s-admissible* if every s consecutive terms are free of nontrivial three-term arithmetic progressions. Freiman's question in the AIM workshop list [1, Problem 1.3] asks for the largest possible density and for the extremal sets. Croot and Lev [2, Section 3.5] record the density question and Freiman's examples of densities $2/3$ for $s = 4$ and $4/9$ for $s = 8$. They also credit Konyagin with upper and lower bounds in terms of the shortest interval containing an s-element progression-free set. We retain these constructions and distinguish finite computation from an explicit closed expression in s.

Translate a_1 to zero and write $g_i = a_{i+1} - a_i \in \mathbb{N}$. Admissibility is equivalent to

$$\sum_{t=j}^{j+p-1} g_t \neq \sum_{t=j+p}^{j+p+q-1} g_t \qquad (j, p, q \geq 1,\ p + q \leq s - 1). \tag{1}$$

Indeed, an equality gives the progression at indices $j, j+p, j+p+q$; conversely every forbidden progression has this form. Any shorter index interval extends to an s-term window in the infinite sequence. A finite gap word is *legal* when it satisfies all instances of (1) contained in that word.

Let $\overline{d}$ and $\underline{d}$ denote upper and lower asymptotic density in the nonnegative integers. The elementary gap identities are

$$\overline{d}(A) = \left(\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g_j \right)^{-1}, \qquad \underline{d}(A) = \left(\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g_j \right)^{-1}, \tag{2}$$

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where $1/\infty = 0$. These follow by evaluating the counting function at sequence terms and immediately before the next terms. Write R_s for the supremum of $\overline{d}(A)$ over all s-admissible A. We will prove that the same maximum is obtained for natural, lower asymptotic and upper Banach density. We use

$$d^*(A) = \limsup_{L\to\infty} \sup_{x\in\mathbb{Z}} \frac{|A \cap [x, x+L)|}{L}.$$

2 A uniform gap cap

Set

$$D_s = \sum_{\ell=2}^{s-1} \ell(\ell-1) = \frac{s(s-1)(s-2)}{3}, \qquad B_s = D_s + 1. \tag{3}$$

Theorem 1 (Pointwise bounded domination)**.** *Every finite legal or infinite admissible gap word g has a legal word b of the same length such that*

$$1 \le b_i \le \min(g_i, B_s), \qquad b_i = g_i \quad \text{if } g_i \le B_s.$$

There is a deterministic such map $T_s : g \mapsto b$, requiring at most $s-2$ positions of look-ahead at each step.

Proof. Process indices from left to right, leaving entries at most B_s unchanged. For an entry at index i greater than B_s, consider each equality forbidden by (1) that involves i. With all other entries at their current values, it is a linear equation in g_i with coefficient 1 or -1. For total length ℓ, at most ℓ intervals contain i, and each has $\ell-1$ splits. Thus at most D_s equations are relevant. Each excludes at most one value. Choose the least integer in $\{1,\dots,B_s\}$ excluded by none of them. This decreases the entry and preserves every constraint, whether or not it contains i.

For an infinite word, every fixed constraint stabilizes after its last index has been processed. It therefore remains valid in the coordinatewise limit. Only indices at distance at most $s-2$ from i occur in the relevant equations. The same procedure for a finite word simply omits intervals crossing an endpoint. □

The choice avoiding the excluded values matters. Hard clamping does not preserve admissibility: for $s=3$, the legal word $(4,5)$ would become $(3,3)$ if both entries were replaced by $B_3 = 3$.

3 Finite optimization and periodic attainment

Put $r = s-2$ and $B = B_s$. Define a weighted directed graph G_s. Its vertices are the legal r-tuples in $\{1,\dots,B\}^r$. There is an edge of weight y from $(x_1,\dots,x_r)$ to $(x_2,\dots,x_r,y)$ precisely when $(x_1,\dots,x_r,y)$ is legal. Thus infinite walks encode exactly the bounded admissible words, apart from their initial r symbols.

The graph has a directed cycle. To see this, the word $1,2,4,8,\dots$ is admissible: its later adjacent block has sum greater than its earlier block. Apply Theorem 1; the resulting infinite walk in a finite graph repeats a vertex. Define

$$\mu_s = \min_C \frac{\sum_{e\in C} w(e)}{|C|}, \tag{4}$$

where C ranges over the nonempty simple directed cycles of G_s.

Theorem 2 (Exact finite characterization)**.** *For every $s \ge 3$, $R_s = 1/\mu_s$. This is the maximum for each of the four density conventions above. It is rational and computable by a finite algorithm. A maximizing periodic gap word has period at most B_s^{s-2}, and the corresponding integer set has period at most B_s^{s-1}.*

Proof. Every finite walk decomposes into directed cycles and a remaining simple path with at most $|V(G_s)|-1$ edges. The cycles have mean weight at least μ_s, and all other weights are positive. Hence the lower average weight of every infinite walk is at least μ_s. For an arbitrary admissible g, Theorem 1 gives $b \le g$ with this property. Equation (2) gives $\overline{d}(A) \le 1/\mu_s$. Repeating a minimum-mean cycle attains equality with natural density. Its number of gaps is at most $|V(G_s)|$ and their sum is at most $B_s|V(G_s)|$, giving the period bounds.

For completeness there is a uniform block estimate. Compress any finite block of n original gaps. Discard its first r compressed entries and apply the same cycle decomposition. With $C_s = B_s(r+|V(G_s)|-1)$, including the case $n<r$, this yields

$$\sum_{j=i}^{i+n-1} g_j \ge \mu_s n - C_s. \tag{5}$$

An integer interval of length L consequently contains at most $L/\mu_s + O_s(1)$ terms. This proves the upper Banach bound; the periodic witness attains it and also attains the natural and lower asymptotic bounds. Finally, the finite graph and its simple cycles can be enumerated. Karp's algorithm [3] is a more efficient classical way to compute (4), componentwise if necessary. □

No complexity claim polynomial in s is intended: even the bound on the number of states is B_s^{s-2}. The bounded-alphabet graph and cycle optimization are standard. The gap cap is the step that justifies passing from an unrestricted integer sequence to this particular finite graph.

4 An exact description of all extremizers

Fix a rational potential $h : V(G_s) \to \mathbb{Q}$ such that every edge has nonnegative reduced cost

$$c(u,v) = w(u,v) - \mu_s + h(u) - h(v) \ge 0. \tag{6}$$

Such a potential exists. Subtract μ_s from every weight, adjoin a source with zero edges to all vertices, and take shortest-path distances. There is no negative cycle, so these distances are finite and satisfy (6).

For any admissible g, let $b = T_s(g)$ be the specific compression of Theorem 1. For $j \ge r+1$ write $v_{j-1} = (b_{j-r}, \dots, b_{j-1})$ and let v_j be the successor obtained by appending b_j. Define

$$\eta_j = (g_j - b_j) + c(v_{j-1}, v_j) \ge 0. \tag{7}$$

The dependence on this fixed compression causes no ambiguity.

Theorem 3 (Extremizer criteria)**.** *With these definitions the following are necessary and sufficient:*

1. $\overline{d}(A) = R_s$ *if and only if* $\liminf_{n\to\infty} n^{-1}\sum_{j=r+1}^{n} \eta_j = 0$.
2. *A has natural density R_s if and only if* $n^{-1}\sum_{j=r+1}^{n}\eta_j \to 0$. *The same condition is equivalent to* $\underline{d}(A) = R_s$.
3. $d^*(A) = R_s$ *if and only if the sequence* $(\eta_j)_{j\ge r+1}$ *has arbitrarily long consecutive runs of zeros.*

All periodic extremizers have gaps at most B_s and correspond exactly to periodic walks using only zero-reduced-cost edges of G_s.

Proof. For $i \geq r+1$, the defining identity telescopes to

$$\sum_{j=i}^{i+n-1} g_j = n\mu_s + \sum_{j=i}^{i+n-1} \eta_j - h(v_{i-1}) + h(v_{i+n-1}). \tag{8}$$

The potential terms are uniformly bounded. Together with (2), this proves the first two assertions. For lower density, the lower gap mean is already at least μ_s; an upper gap mean of μ_s is therefore equivalent to convergence to it.

Long zero-defect runs give blocks whose span is $n\mu_s + O_s(1)$, hence upper Banach density at least $1/\mu_s$. Conversely, if this density is attained, take longer and longer integer intervals with counts divided by lengths approaching $1/\mu_s$. Their internal gap blocks have lengths tending to infinity and, by (5), mean tending to μ_s. Finitely many initial indices may be discarded. Equation (8) makes the corresponding mean defect tend to zero. All positive defects are bounded below by a fixed $\epsilon_s > 0$: the compression excess is an integer, and the edge costs form a finite set of nonnegative rationals. If zero runs were shorter than L, every L consecutive defects would contribute at least ϵ_s, a contradiction.

If a periodic extremizer had a gap greater than B_s, compression would decrease its average by at least the positive frequency of that gap. The resulting legal word would have lower gap mean less than μ_s, contradicting Theorem 2. For a bounded periodic word the compression excess vanishes. Summing the nonnegative costs over one period shows that its mean is μ_s exactly when every edge cost is zero. $\square$

For natural density the condition can also be read as $\sum_{j \leq n}(g_j - b_j) = o(n)$ and only $o(n)$ positive-cost edges. For upper density both must be small along the *same* subsequence; separate lower-limit conditions do not suffice. Counting changed gaps without their sizes likewise does not suffice.

Proposition 4. *For every $s \geq 3$ there are natural-density extremizers with unbounded gaps.*

Proof. Start with a minimizing periodic gap word, bounded by B_s. At indices $n_k = 2s\,2^k$, $k \geq 1$, replace its gap by $sB_s + k$. Any constraint (1) meets at most one modified index. That gap exceeds the sum of all its at most $s-2$ companions, so it cannot participate in an equality. The added length among the first n gaps is $O_s((\log n)^2) = o(n)$. Thus the mean and the positive natural density are unchanged, while gaps are unbounded. $\square$

5 The sharp values for five through eight terms

Lemma 5. *For a legal triple put*

$$H(a,b,c) = \min\{0,\ 4c-9,\ 4(b+c)-18,\ 4(a+b+c)-27\}.$$

Every legal quadruple (a,b,c,d), with no upper bound on its positive integer entries, satisfies

$$4d - 9 + H(a,b,c) - H(b,c,d) \geq 0. \tag{9}$$

Proof. If one of the first three candidates attains $H(a,b,c)$, adding $4d-9$ gives a candidate for $H(b,c,d)$, so the inequality follows. If only the last candidate is minimal, the resulting expression before subtracting $H(b,c,d)$ is $4(a+b+c+d)-36$. This is nonnegative if $a+b+c+d \geq 9$, whereas $H(b,c,d) \leq 0$.

The legal quadruples with total at most eight are exactly

$$(1,2,4,1), \quad (1,4,1,2), \quad (1,4,2,1), \quad (2,1,4,1). \tag{10}$$

Here is a small direct exhaustion. Adjacent entries must differ. An entry at least five forces the other three to have total at least four. If an entry four is at an end, the remaining triple must be $(1, 2, 1)$, making four equal to its sum. An internal four leaves exactly the four possibilities in (10). If all entries are at most three and no three occurs, the entries alternate one and two, giving equal adjacent pair sums. With one three, if it is internal, it neighbors a pair one and two on one side; if it is at an end, the remaining triple alternates one and two. Either way a forbidden equality $3 = 1 + 2$ occurs. With two threes, the other entries must both be one and alternate with them, again giving equal adjacent pair sums. More threes exceed the total bound or give adjacent equality.

For the four legal words in (10), the values of $H(a, b, c)$ are $0, -5, -1, 0$, while the respective last candidates are $1, -3, 1, 1$. Thus the last candidate is never minimal in these remaining cases. This proves (9). □

Theorem 6. *The maximum density is* $R_3 = R_4 = 2/3$ *and*

$$R_5 = R_6 = R_7 = R_8 = 4/9.$$

These values hold for all four density conventions used above.

Proof. On legal triples $-11 \le H \le 0$: the last entry is at least one, the last pair sums to at least three, and the triple sums to at least four. Sum (9) along any five-admissible gap word. The potential telescopes, giving lower mean gap at least $9/4$. Consequently $R_5 \le 4/9$.

Freiman's source example is $\{0, 1, 3, 4\} + 9\mathbb{Z}_{\ge 0}$, with gap period $(1, 2, 1, 5)$ and density $4/9$. The only solutions of $x + z = 2y$ modulo nine within $\{0, 1, 3, 4\}$ have $x = y = z$. Thus any progression in this periodic set has equal residues, and its first and last terms have sequence indices differing by at least eight. It cannot occur in eight consecutive terms. This proves $R_8 \ge 4/9$; monotonicity in s proves the four equalities. For $s = 3, 4$, consecutive gaps are distinct, hence their sum is at least three. The source period $(1, 2)$ is four-admissible and has mean $3/2$. Theorem 2 transfers these maxima to the other density conventions. □

6 Scope and reproducibility

This is an effective answer for every fixed s, with exact necessary-and-sufficient extremizer criteria. It does not provide a short formula in s, a sharp asymptotic, or the smallest gap cap. The periodicity assertion is existence of a periodic optimum, not eventual periodicity of all extremizers, as Proposition 4 demonstrates.

The frozen research aid for AIM-COMBINATORICS-0209 [4] already contains the gap encoding, the $s = 3, 4$ values, the source constructions and optimization under an imposed gap bound. It explicitly leaves a universal gap reduction and the $s = 5$ optimum unproved. Theorem 1, Theorem 3 and Lemma 5 supply those additional arguments. We make no novelty claim for the source examples, finite-state optimization, shortest-path potentials or Karp's algorithm. The retained bounded literature search is not certification of absolute priority.

The accompanying standard-library Python checker reconstructs the full capped graphs for $s = 3, 4, 5$ with an independent point-triple predicate, verifies every dual edge inequality and a matching cycle, and checks finite compression and boundary examples. The respective graph sizes $(|V|, |E|)$ are $(3, 6)$, $(72, 512)$ and $(8000, 145208)$. Both Python 3.9.6 and 3.13.5 pass 199,510 assertions with identical mathematical reports. These finite checks are diagnostics, not a replacement for the general proofs above.

This unrefereed manuscript was developed with AI assistance for derivation, literature retrieval, computation and exposition. The author is responsible for its claims. The originating researcher self-audit is not independent review or formal verification.

References

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[2] E. S. Croot III and V. F. Lev, Open problems in additive combinatorics, in *Additive Combinatorics*, CRM Proceedings and Lecture Notes **43**, American Mathematical Society (2007), 207–233, Section 3.5. Author manuscript: `https://math.haifa.ac.il/seva/Papers/Montpr.pdf`.

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[4] ulamai, *UnsolvedMath*, version 1.6.0, record AIM-COMBINATORICS-0209, frozen snapshot `c6e7f41b4eb6`. Hugging Face dataset.