

A Sharp Three-Fifths Bound for Seymour Vertices in Three-Regular Oriented Graphs

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Abstract

Let D be a finite simple oriented graph in which every vertex has indegree and outdegree three. A vertex is a Seymour vertex if its exact second out-neighborhood is at least as large as its first. We prove that at least $\lceil 3|V(D)|/5 \rceil$ vertices are Seymour vertices. If B is the set of the remaining vertices, then $\sum_v (|N^{++}(v)| - 3) \geq \lceil |B|/2 \rceil$. Both constants are sharp, even for strongly connected graphs. We classify all graphs attaining the unrounded three-fifths proportion: they arise from loopless two-in/two-out directed multigraphs by replacing vertices with directed triangles and arcs with vertices joined completely to the two endpoint triangles. This representation gives an exact surplus formula and characterizes simultaneous equality in both bounds. The proof uses the forced local structure of a deficient vertex and indegree saturation. These results concern the three-regular class; they do not resolve the mean second-neighborhood inequality for arbitrary Eulerian oriented graphs. This manuscript is unrefereed.

1 Notation and main result

All oriented graphs in this paper are finite and simple, with no loops or pairs of opposite arcs. We write $u \rightarrow v$ for an arc. For a vertex v , let $N^+(v)$ be its out-neighborhood and let

$$N^{++}(v) = \left(\bigcup_{u \in N^+(v)} N^+(u) \right) \setminus (N^+(v) \cup \{v\}).$$

Thus $N^{++}(v)$ consists of vertices at directed distance exactly two, not all endpoints of two-step walks. A *Seymour vertex* satisfies $|N^{++}(v)| \geq |N^+(v)|$. We call D *three-regular* when $d^+(v) = d^-(v) = 3$ at every vertex. This is not a condition on total undirected degree three.

For a three-regular oriented graph, put $n = |V(D)|$ and define

$$\begin{aligned} q(v) &= |N^{++}(v)|, & B &= \{v : q(v) < 3\}, \\ P &= \{v : q(v) > 3\}, & Z &= \{v : q(v) = 3\}, \\ F(D) &= \sum_{v \in V(D)} (q(v) - 3). \end{aligned}$$

The sets B, P, Z partition the vertex set, and $SV(D) = P \cup Z$ is the set of Seymour vertices.

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Theorem 1.1. *Every three-regular oriented graph satisfies*

$$3|B| \leq 2|P|, \quad |\text{SV}(D)| \geq \left\lceil \frac{3n}{5} \right\rceil, \quad F(D) \geq \left\lceil \frac{|B|}{2} \right\rceil. \quad (1)$$

The constants $3/5$ and $1/2$ are sharp even when D is strongly connected.

The result is related to, but distinct from, the general Eulerian mean question recorded by Sullivan [6, Conjecture 6.15] and Cary [2, Conjecture 1.1]:

$$\sum_v |N^{++}(v)| \geq \sum_v d^+(v) \quad \text{when } d^+(v) = d^-(v) \text{ for every } v. \quad (2)$$

We use the simple oriented-graph domain throughout. Theorem 1.1 implies (2) in the three-regular class, and strengthens it when at least one vertex is not Seymour. It does not prove (2) for varying degrees at most three, or for arbitrary Eulerian oriented graphs.

Existence of a Seymour vertex under small minimum-outdegree assumptions is classical; see Kaneko and Locke [4] and the background in [1, 2]. Mody [5, Theorem 5.1] already proves the Eulerian mean inequality for maximum degree at most two. Cary proves an all-vertices result under a simple cycle-intersection condition [2, Theorem 2.7]. The number of Seymour vertices in random tournaments and digraphs is studied in [3]. Our subject is a deterministic sharp proportion and its equality cases in the three-regular class. The arguments below are self-contained. No absolute priority claim is made for the elementary local method or the resulting statements.

2 The neighborhood of a deficient vertex

Lemma 2.1. *Let $x \in B$, and put $A = N^+(x)$ and $C = N^{++}(x)$. Then $|C| = 2$, $D[A]$ is a directed triangle, and every vertex of A sends an arc to both vertices of C . Each vertex of C has all its predecessors in A and all its successors outside $A \cup C$. Moreover, $A \subseteq P$.*

Proof. There are nine arcs leaving the three vertices of A . Every one ends in $A \cup C$: its endpoint is reached in two steps from x , and cannot be x because digons are excluded. There are at most three arcs within A and at most $3|C|$ arcs from A to C . Since $|C| \leq 2$,

$$9 \leq 3 + 3|C| \leq 9.$$

Equality forces $|C| = 2$, all six arcs from A to C , and exactly three arcs within A . Each vertex of A already sends two arcs to C , so its remaining outgoing arc is internal. Thus $D[A]$ is a directed triangle.

Every $z \in C$ receives three arcs from A , saturating its indegree. There is no arc within C , because its head would then have indegree at least four. There is no arc from C to A , because all arcs from A to C are present and D has no digons. Hence $N^+(z) \cap (A \cup C) = \emptyset$ for every $z \in C$.

Write the internal triangle as $a \rightarrow a' \rightarrow a'' \rightarrow a$. Its vertex a'' is an exact second neighbor of a . Fix either $z \in C$. All three vertices of $N^+(z)$ are also exact second neighbors of a : they are outside $A \cup C$, whereas $N^+(a) = \{a'\} \cup C$. They are distinct from a'' . Thus $q(a) \geq 4$. The same argument applies to all three members of A , proving $A \subseteq P$. $\square$

Proof of the inequalities in Theorem 1.1. Every outgoing arc of a vertex in B ends in P , by Lemma 2.1. If $a \in P$ has a predecessor $x \in B$, then a belongs to the directed triangle $D[N^+(x)]$. Its predecessor on that triangle lies in P as well. Since $d^-(a) = 3$, at most two predecessors of a

can belong to B . The same upper bound is trivial if it has no predecessor in B . Counting arcs from B to P gives

$$3|B| \leq 2|P| \leq 2(n - |B|).$$

Consequently $|B| \leq \lfloor 2n/5 \rfloor$ and $|\text{SV}(D)| = n - |B| \geq \lceil 3n/5 \rceil$. Every vertex in B has deficit exactly one, by Lemma 2.1, and every vertex in P has surplus at least one. Therefore

$$F(D) \geq |P| - |B| \geq \frac{|B|}{2}.$$

Taking the ceiling is valid because $F(D)$ is an integer. $\square$

3 Sharpness

Construction 3.1. For an integer $k \geq 2$, take disjoint sets A_i of size three and B_i of size two, with indices modulo k . Orient each A_i as a directed triangle and leave each B_i independent. Add all arcs $A_i \rightarrow B_i$ and all arcs $B_i \rightarrow A_{i+1}$, and no others. Denote the resulting graph by D_k .

The condition $k \geq 2$ prevents digons. A vertex of A_i has one internal and two external predecessors and successors. A vertex of B_i has three predecessors in A_i and three successors in A_{i+1} . Thus D_k is three-regular. Following the cyclic layers, and using the internal triangles, shows that D_k is strongly connected.

For $a \in A_i$, its exact second neighborhood consists of the third vertex of the internal triangle and all of A_{i+1} , so $q(a) = 4$. Two-step walks ending in B_i only reach existing first neighbors. For $b \in B_i$, its exact second neighborhood is B_{i+1} , so $q(b) = 2$. Therefore

$$n = 5k, \quad |B| = 2k, \quad |\text{SV}(D_k)| = 3k, \quad F(D_k) = k. \quad (3)$$

This proves sharpness in Theorem 1.1. In particular the class contains arbitrarily large strongly connected graphs in which two fifths of the vertices are not Seymour vertices.

4 Classification at the exact proportion

The equality considered next is $|\text{SV}(D)| = 3n/5$ without rounding. It requires $5 \mid n$. We do not assert that the classification includes every graph attaining the rounded lower bound when $5 \nmid n$.

Construction 4.1. Let H be a finite nonempty loopless directed multigraph whose indegrees and outdegrees, counting multiplicity, are all two. Parallel arcs and pairs of opposite arcs in H are allowed. Replace each vertex i of H by a directed triangle A_i . For each arc $e : i \rightarrow j$ of H , introduce a new vertex b_e , add all arcs $A_i \rightarrow b_e$ and $b_e \rightarrow A_j$, and add no other arcs. Write $D(H)$ for the resulting oriented graph.

Theorem 4.2. *For a nonempty three-regular oriented graph D , the following are equivalent:*

1. $|\text{SV}(D)| = 3|V(D)|/5$;
2. D is isomorphic to $D(H)$ for some multigraph H as in Construction 4.1.

In this representation the A_i are exactly the components of $D[P]$, and the b_e are exactly B . Thus H is determined by D up to isomorphism, including arc multiplicities.

Proof. Assume the first assertion, equivalently $5|B| = 2n$. Both inequalities in

$$3|B| \leq 2|P| \leq 2(n - |B|)$$

are equalities. Hence Z is empty and every vertex of P has exactly two predecessors in B and one in P . Every member of P lies in $N^+(b)$ for some $b \in B$, which is a directed triangle by Lemma 2.1. Two such triangles sharing a vertex must coincide: its predecessor in P is unique, and applying the same fact at that predecessor identifies the third vertex. These triangles consequently partition P .

Fix a triangle $A = N^+(b)$. Both vertices of $N^{++}(b)$ receive all three incoming arcs from A . They cannot belong to P , since every member of P has two predecessors in B . Since Z is empty, they belong to B . Lemma 2.1 now describes every outgoing arc of a vertex of A : one internal arc and two arcs to this common pair in B . In particular $D[P]$ has no additional arcs between the triangles.

Every $b \in B$ sends its three outgoing arcs to one whole triangle. There are no arcs between bad vertices. Choose a predecessor $a \in P$ of b . All three members of a 's triangle send to b , and these arcs saturate b 's indegree. Thus b has a unique predecessor triangle and a unique successor triangle. They are distinct, since otherwise digons would occur.

Contract each triangle to a vertex, and represent each $b \in B$ by an arc from its predecessor triangle to its successor triangle. Each triangle has two outgoing bad vertices. It also has two incoming bad vertices: a bad predecessor of one triangle vertex sends to the whole triangle, and each triangle vertex has exactly two bad predecessors. The resulting multigraph H is loopless and has indegree and outdegree two. This recovers every arc of D exactly as in Construction 4.1.

Conversely, let H have t vertices and $2t$ arcs. The graph $D(H)$ is simple because different arcs of H have distinct new vertices; it has no digons because H has no loops. Each triangle vertex has one internal and two external predecessors and successors, and each arc vertex has three of each. Hence $D(H)$ is three-regular on $5t$ vertices.

For an arc vertex b_e with head j , two-step walks through A_j either end at an existing first neighbor or at one of the two arc vertices leaving j . These two vertices are distinct, neither is b_e , and neither belongs to A_j . Thus $q(b_e) = 2$.

Let r_i be the number of distinct out-neighbors of i in H . It is either one or two. For $a \in A_i$, its exact second neighborhood consists of the third vertex of A_i and all three vertices of each A_j for which $i \rightarrow j$ is an arc of H . All other two-step walks end at existing first neighbors. Since H has no loops, these sets are disjoint and

$$q(a) = 1 + 3r_i \in \{4, 7\}. \quad (4)$$

Precisely the $2t$ arc vertices are bad, and the $3t$ triangle vertices are strictly good. This proves the first assertion and also the intrinsic description of the two parts. Recovering each arc vertex's predecessor and successor triangles proves the stated uniqueness of H . $\square$

5 Surplus and simultaneous equality

Proposition 5.1. *Let H have t vertices, and let s be the number of its vertices with two distinct out-neighbors. Then*

$$F(D(H)) = t + 9s. \quad (5)$$

Moreover, $D(H)$ is strongly connected if and only if H is strongly connected.

Proof. By (4), the three vertices of A_i contribute $3(3r_i - 2)$ to the surplus. Each of the $2t$ arc vertices contributes -1 . Since $\sum_i r_i = t + s$,

$$F(D(H)) = \sum_i (9r_i - 6) - 2t = 9(t + s) - 8t = t + 9s.$$

Directed paths in H lift through the indicated triangles and arc vertices. The directed cycle inside each triangle connects its vertices, and every arc vertex has the prescribed incoming and outgoing connections. Thus strong connectivity of H implies that of $D(H)$. Conversely, a path between triangles in $D(H)$ projects, after ignoring internal triangle steps, to a directed walk in H . Strong connectivity of $D(H)$ therefore implies that of H . $\square$

Corollary 5.2. *A nonempty three-regular oriented graph satisfies both*

$$|\text{SV}(D)| = \frac{3n}{5} \quad \text{and} \quad F(D) = \frac{|B|}{2}$$

if and only if it is a disjoint union of graphs D_k from Construction 3.1, with $k \geq 2$ in each component.

Proof. Theorem 4.2 represents D as $D(H)$ with $n = 5t$ and $|B| = 2t$. Proposition 5.1 shows that surplus equality holds exactly when $s = 0$. In that case the two outgoing arcs at every vertex of H are parallel. Their common-target map is a permutation: the indegree-two condition permits exactly one predecessor vertex at each target. The permutation has no fixed points because H is loopless. Thus H is a disjoint union of directed cycles of length at least two, with every arc doubled. Expanding these cycles gives precisely the graphs in Construction 3.1. Their equalities were verified in (3). $\square$

Remark 5.3. The two equality requirements are distinct. Let H have three vertices and one arc in each direction between every pair. Then $t = s = 3$, so $D(H)$ has fifteen vertices, exactly nine Seymour vertices, and $F(D(H)) = 30$, whereas $|B|/2 = 3$. Equality in the vertex proportion alone does not force the smallest possible total surplus.

6 Scope and reproducibility

The proofs above use only local arc capacities, exact neighborhood counts and equality in the resulting incidence bound. No computational enumeration is required for the general statements. The accompanying standard-library Python checker supplies regression tests using explicit arc sets and neighborhood bit masks. It verifies all 916 loopless two-in/two-out multigraphs on two through five labelled vertices (1, 3, 42, 870 respectively), their expansions and a random relabelling of each. It also checks 300 larger expansions, another relabelling of each, 2400 nonuniform degree-preserving-switch samples of three-regular oriented graphs, and four invalid-input controls. Normal and optimized Python runs give identical outputs; the checker uses explicit exceptions rather than assertions that optimization removes. The earlier sharp-family checker separately checks D_k for $2 \leq k \leq 20$ and rejects the invalid $k = 1$ case.

These tests are finite error-detection checks, not formal verification or independent mathematical review. The multigraph enumeration is exhaustive only in its stated two-in/two-out domain; the switched three-regular sample is neither uniform nor exhaustive. The statement about all equality cases rests on the proof of Theorem 4.2.

The exact-proportion classification does not classify every equality case of $F(D) = \lceil |B|/2 \rceil$ in isolation. The unrestricted Eulerian mean question (2), including mixed degrees at most three, is

left open here. In the associated problem inventory this paper is a complete special-class result, not a closure of the whole source record.

AI assistance was used in developing arguments, searching the literature, writing and running verification code, and preparing the manuscript. The stated proofs have been self-audited; the manuscript has not undergone independent peer review or proof-assistant verification.

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