

# Cyclic Counterexamples to the Exact-Distance Higher-Neighborhood Conjecture

Alper Ferudun\*

18 October 2026

## Abstract

For every integer  $k \geq 3$  and  $n > k(k+1)$ , the oriented Cayley graph  $\text{Cay}(\mathbb{Z}/n\mathbb{Z}, \{1, k, k+1\})$  has directed girth greater than  $k$ , but every vertex has  $2k-1$  vertices at shortest directed distance  $k-1$  and only  $k+1$  at distance  $k$ . This refutes the exact-distance higher-neighborhood conjecture printed as Conjecture 6.13 in Sullivan’s 2006 survey. The examples are strongly connected, vertex-transitive, and three-in/three-out regular; their directed girth can be arbitrarily large at each fixed  $k$ . Independent blow-ups give unbounded regular degrees and shell deficits. The proof is an elementary interval identity for integer sumsets. A thirteen-vertex example illustrates both the counterexample and the distinction between exact-distance neighborhoods and a different external-neighborhood recursion. The second-neighborhood case  $k = 2$  is not resolved.

## 1 The assertion and the construction

For a finite digraph  $D$ , let  $d_D(v, w)$  be the length of a shortest directed path from  $v$  to  $w$ , with value  $\infty$  if no such path exists. Put

$$N_r^+(v) = \{w : d_D(v, w) = r\}, \quad B_r^+(v) = \{w : d_D(v, w) \leq r\}.$$

In particular  $B_0^+(v) = \{v\}$ . The directed girth  $g(D)$  is the minimum length of a directed cycle. An oriented graph has no loops, parallel arcs, or antiparallel pair of arcs.

Conjecture 6.13 in Sullivan’s survey [1], attributed there to an unpublished manuscript of Serge Burckel, asserts that absence of directed cycles of length at most  $k$  guarantees a vertex satisfying

$$|N_k^+(v)| \geq |N_{k-1}^+(v)|. \tag{1}$$

The survey defines  $N_r^+$  by shortest distance exactly  $r$  on page 1 and prints the conjecture on page 8. We address that printed formulation, without making an assertion about the unrecovered unpublished manuscript. The case  $k = 2$  is the second-neighborhood question; our counterexamples concern every  $k \geq 3$ .

**Theorem 1.** *Let  $k \geq 3$  and  $n > k(k+1)$  be integers. Define  $D_{k,n}$  on  $\mathbb{Z}/n\mathbb{Z}$  by the arcs*

$$x \longrightarrow x+1, \quad x \longrightarrow x+k, \quad x \longrightarrow x+k+1.$$

*This is a strongly connected, vertex-transitive oriented graph with indegree and outdegree three. It satisfies*

$$g(D_{k,n}) \geq \left\lceil \frac{n}{k+1} \right\rceil > k, \tag{2}$$

---

\*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: AlperTheKing.

and, at every vertex  $v$ ,

$$|N_{k-1}^+(v)| = 2k - 1, \quad |N_k^+(v)| = k + 1 < 2k - 1. \quad (3)$$

For  $n = k(k + 1) + 1$  its directed girth is exactly  $k + 1$ .

Thus (1) fails at every vertex, not merely at one chosen root. Letting  $n$  grow while holding  $k$  fixed shows that a larger lower bound on directed girth alone cannot repair the assertion.

## 2 An integer ball calculation

Fix  $k \geq 2$  and let  $A_r$  denote the set of integers expressible as a sum of at most  $r$  members of  $\{1, k, k + 1\}$ , including the empty sum.

**Lemma 2.** *For every integer  $r \geq 0$ ,*

$$A_r = \bigcup_{j=0}^r ([kj, kj + r] \cap \mathbb{Z}). \quad (4)$$

Consequently,

$$|A_r| = \begin{cases} (r + 1)^2, & 0 \leq r \leq k - 1, \\ (k + 1)r + 1, & r \geq k - 1. \end{cases} \quad (5)$$

The two formulas agree at  $r = k - 1$ .

*Proof.* Group a sum by the number  $j$  of summands equal to  $k$  or  $k + 1$ . If exactly  $b$  of those are  $k + 1$  and  $i$  further summands are 1, the total is  $kj + b + i$ , where

$$0 \leq b \leq j, \quad 0 \leq i \leq r - j.$$

For fixed  $j$ , the values of  $b + i$  fill all of  $[0, r] \cap \mathbb{Z}$ : for  $u \leq j$  take  $b = u, i = 0$ , and for  $u > j$  take  $b = j, i = u - j$ . This proves (4) in both directions.

If  $r \leq k - 1$ , consecutive intervals have disjoint integer points, so the  $r + 1$  intervals each contribute  $r + 1$  points. If  $r \geq k - 1$ , there is no integer gap between consecutive intervals, and their union is  $[0, (k + 1)r] \cap \mathbb{Z}$ . This proves (5).  $\square$

*Proof of Theorem 1.* For  $0 \leq r \leq k$ , every element of  $A_r$  lies in  $[0, k(k + 1)]$ . Reduction modulo  $n$  is injective on this interval, and the image of  $A_r$  is exactly  $B_r^+(0)$ . A walk of at most  $r$  steps gives such a sum; conversely such a sum gives a walk and hence a path of length at most  $r$ . The lemma gives

$$|A_{k-2}| = (k - 1)^2, \quad |A_{k-1}| = k^2, \quad |A_k| = k(k + 1) + 1.$$

Subtracting successive ball sizes proves (3) at zero. Translation is a graph automorphism, so these counts hold at every vertex.

A nonempty directed closed walk of length  $\ell$  has a positive integer total increment divisible by  $n$ . That total is at most  $\ell(k + 1)$ . Hence  $\ell(k + 1) \geq n$ , proving (2). The three increments are distinct and nonzero modulo  $n$ ; thus both degrees are three and no parallel arcs occur. The girth bound excludes loops and digons. The increment 1 provides a directed Hamilton cycle, proving strong connectivity.

Finally, if  $n = k(k + 1) + 1$ , take  $k$  successive increments of  $k + 1$ , followed by one increment of 1. All intermediate integer totals are distinct and smaller than  $n$ . This gives a directed cycle of length  $k + 1$ , matching the lower bound.  $\square$

The proof uses balls of radius at most  $r$ . Counting only walks of length exactly  $r$  would give a different statement. The strict condition  $n > k(k + 1)$  is also essential to the no-wrap argument.

### 3 A small witness and two extensions

For  $k = 3$  and  $n = 13$ , the increments are 1, 3, 4. Directly at zero,

$$\begin{aligned} N_1^+(0) &= \{1, 3, 4\}, \\ N_2^+(0) &= \{2, 5, 6, 7, 8\}, \\ N_3^+(0) &= \{9, 10, 11, 12\}. \end{aligned}$$

The cycle  $0 \rightarrow 4 \rightarrow 8 \rightarrow 12 \rightarrow 0$  has length four, and the positive-increment argument excludes any shorter directed cycle. This supplies an explicit thirteen-vertex witness; we make no claim that its order is globally minimal. Notice also that every vertex satisfies the second-neighborhood inequality  $5 \geq 3$ .

**Corollary 3.** *For each  $k \geq 3$  and each integer  $g \geq 1$ , there is a strongly connected, vertex-transitive, three-in/three-out oriented graph of directed girth at least  $g$  in which (1) fails at every vertex.*

*Proof.* Choose an integer  $n > \max\{k(k+1), (g-1)(k+1)\}$  and apply the theorem.  $\square$

**Corollary 4.** *For every  $k \geq 3$ ,  $n > k(k+1)$ , and integer  $m \geq 1$ , there is a strongly connected, vertex-transitive oriented graph with both degrees  $3m$ , directed girth  $g(D_{k,n})$ , and shell sizes  $m(2k-1)$  and  $m(k+1)$  at distances  $k-1$  and  $k$  at every vertex. Its strict shell deficit is  $m(k-2)$ .*

*Proof.* Replace each vertex of  $D_{k,n}$  by an independent set of size  $m$  and every base arc by all arcs from its source part to its target part. The degree, orientation, transitivity, and connectivity claims follow directly. A directed cycle projects to a nonempty closed walk of the same length in the base. Every such closed walk contains a directed cycle, so its length is at least the base girth. Conversely a shortest base cycle lifts. The girths agree.

Fix a root in one part. No other vertex of that part is reachable in at most  $k$  steps: such a path would project to a base closed walk of length at most  $k$ . For every other part at base distance  $r$ , all its vertices are at distance exactly  $r$  from the root. It follows that each shell of positive radius at most  $k$  has  $m$  times its base cardinality. The theorem gives the result.  $\square$

The examples contain the transitive triangle on  $0, 1, k+1$ . Their large directed girth should therefore not be confused with large girth of the underlying undirected graph.

### 4 Neighborhood conventions and verification

For a vertex set  $X$ , consider the external-neighborhood operator

$$F(X) = \{y : x \rightarrow y \text{ for some } x \in X\} \setminus X.$$

Daamouch [2, p. 156] uses this definition and iterates it in the higher-order discussion on page 161. Such iteration need not give exact-distance shells: it removes the immediately preceding set, not all previously visited vertices.

Indeed, in  $D_{3,13}$ , put  $X_0 = \{0\}$  and  $X_{r+1} = F(X_r)$ . The first two iterates equal  $N_1^+(0)$  and  $N_2^+(0)$ , but

$$X_3 = \{3, 9, 10, 11, 12\}, \quad |X_3| = |X_2| = 5.$$

The vertex 3 has returned from an earlier layer. Thus the displayed witness does not contradict the corresponding recursive inequality. We make no claim to refute that different formulation.

Similarly, the endpoint sets of unrestricted walks of lengths two and three from zero have sizes six and nine, respectively, rather than five and four.

The accompanying standard-library Python checker performs 3,546 exact regression checks in each of normal and optimized mode. It compares the integer identity for  $2 \leq k \leq 40$ ; graph distances for  $3 \leq k \leq 25$  and three moduli per  $k$ ; all-root breadth-first and all-pairs distances on the thirteen-vertex witness; and independent blow-ups for  $1 \leq m \leq 5$ . It also checks the distinct neighborhood conventions and the  $k = 2, n = 7$  equality control. These finite tests supplement, but do not replace, the general proof above.

This note was prepared with AI assistance and subjected to the author’s proof and source audit. It is an unrefereed preprint, not a formally verified proof or a report of independent review. The source comparison was bounded; no absolute priority claim is made.

## References

- [1] B. D. Sullivan, *A Summary of Results and Problems Related to the Caccetta–Haggkvist Conjecture*, AIM report, 14 April 2006. Official AIM PDF; arXiv:math/0605646.
- [2] M. Daamouch, *A note on the second neighborhood problem*, *Utilitas Mathematica* **123** (2025), 155–164. doi:10.61091/um123-12.