A Six-Vertex Counterexample to Vertex-Averaged Ordered Reachability

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Abstract

We give two loopless directed graph layers on six vertices, each with constant outdegree two, having no rainbow directed cycle and hence no increasing rainbow cycle. For the fixed label order $1 < 2$, the sizes of the sets reachable by possibly trivial increasing paths are $5, 5, 5, 5, 5, 4$. Their average is $29/6 < 5 = 1 + \delta^+(G_1) + \delta^+(G_2)$. Thus the uniform-start-vertex interpretation of the ordered-reachability conjecture attributed to DeVos in Sullivan's 2006 survey is false. Uniform blow-ups give examples on $6m$ vertices with average $1 + 23m/6$, below the proposed bound $1 + 4m$ for every $m \geq 1$. The proof is an explicit neighborhood calculation and does not depend on the numerical search that located the example.

1 The precise inequality

Let $G_i = (V, E_i)$, $1 \leq i \leq k$, be simple directed graphs on a common finite nonempty vertex set. An arc may belong to several layers: its admissible labels are those i for which it lies in E_i. This is the convention in [1, Conjecture 6.1]. Simplicity forbids repeated arcs with the same ordered endpoints, not antiparallel arcs [1, p. 1]. A path is *increasing* if its chosen labels strictly increase in the fixed order $1 < \cdots < k$. Write $\mathcal{R}(v)$ for the set of endpoints of such paths starting at v, including the empty path, and put $\delta_i = \min_{v \in V} |N_i^+(v)|$.

Conjecture 6.10 of [1, p. 7], attributed there to DeVos, proposes an average reachability bound in the absence of an increasing rainbow cycle. Its wording combines an average with a fixed starting vertex without explicitly specifying the averaging variable. We address the uniform average over starting vertices, with the label order held fixed:

$$\frac{1}{|V|} \sum_{v \in V} |\mathcal{R}(v)| \;\geq\; 1 + \sum_{i=1}^{k} \delta_i. \tag{1}$$

This convention is part of our statement. We do not identify it with an average over label orders, nor claim to resolve other possible interpretations of the source's sentence. The source records a positive result for Cayley layers on a common group; our layers are not of that form.

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2 The counterexample

Theorem 1. *There are two loopless simple digraph layers G_1, G_2 on six vertices, each of constant outdegree two, with no rainbow directed cycle, for which*

$$\sum_{v \in V} |\mathcal{R}(v)| = 29 < 30 = |V|(1 + \delta_1 + \delta_2)$$

in the label order $1 < 2$.

Proof. Take $V = \{0, 1, 2, 3, 4, 5\}$ and define the layers by the two middle columns of Table 1. They have no loops or repeated arcs, and each vertex has two outgoing arcs in each layer.

With only two labels a rainbow cycle could only be a loop or a digon whose arcs can receive different labels. There are no loops, and direct inspection of the table gives

$$E_1 \cap \{(y, x) : (x, y) \in E_2\} = \varnothing. \tag{2}$$

Consequently there is no rainbow directed cycle. Monochromatic digons are permitted and do occur, for example $0 \leftrightarrow 1$ in layer 1.

Every nontrivial increasing path has one edge or has a label-1 edge followed by a label-2 edge. Hence its endpoint set is exactly

$$\mathcal{R}(v) = \{v\} \cup N_1^+(v) \cup N_2^+(v) \cup \bigcup_{w \in N_1^+(v)} N_2^+(w). \tag{3}$$

The final column of the table evaluates this formula. Five rows have size five and the last row has size four. Their sum is 29, whereas both minimum outdegrees equal 2, giving the claimed strict inequality. $\square$

v	$N_1^+(v)$	$N_2^+(v)$	$\mathcal{R}(v)$ for $1 < 2$
0	$\{1, 2\}$	$\{2, 4\}$	$\{0, 1, 2, 4, 5\}$
1	$\{0, 5\}$	$\{4, 5\}$	$\{0, 1, 2, 4, 5\}$
2	$\{1, 3\}$	$\{1, 5\}$	$\{1, 2, 3, 4, 5\}$
3	$\{0, 4\}$	$\{1, 4\}$	$\{0, 1, 2, 3, 4\}$
4	$\{2, 5\}$	$\{0, 2\}$	$\{0, 1, 2, 4, 5\}$
5	$\{0, 4\}$	$\{0, 2\}$	$\{0, 2, 4, 5\}$

Table 1: The complete six-vertex certificate. Each ordered pair is one arc; an arc in both middle columns has both labels.

3 An infinite family and the role of the order

Corollary 2. *For every integer $m \geq 1$ there are two loopless simple layers on $6m$ vertices with constant outdegrees $2m$, no rainbow directed cycle, and average increasing reachability $1 + 23m/6$ in the order $1 < 2$.*

Proof. Replace every vertex v of the example by an independent set $V_v = \{(v, a) : 1 \leq a \leq m\}$. Replace an arc $v \to w$ in layer i by all m^2 arcs from V_v to V_w in that layer. Each new vertex has outdegree $2m$ in each layer. A rainbow digon would project to a mixed-label digon in the original example; a loop is impossible. Thus there is still no rainbow cycle.

Every nontrivial increasing path projects to an increasing walk in the base graph. No such walk returns to its starting base vertex by (2). Conversely every base path to a different endpoint lifts to any chosen vertex above that endpoint. Therefore

$$\mathcal{R}((v,a)) = \{(v,a)\} \cup \bigcup_{w \in \mathcal{R}(v) \setminus \{v\}} V_w, \qquad |\mathcal{R}((v,a))| = 1 + m(|\mathcal{R}(v)| - 1).$$

There are $5m$ vertices with reachability $1 + 4m$ and m with reachability $1 + 3m$. The average is consequently

$$\frac{5m(1 + 4m) + m(1 + 3m)}{6m} = 1 + \frac{23m}{6} < 1 + 4m.$$

□

The label order is essential. For the same six-vertex example with $2 < 1$, the reachability sizes are $6, 5, 6, 6, 6, 6$, totaling 35. Averaging also over the two label orders gives $(29+35)/12 = 16/3 > 5$, so this example is not a counterexample to that different averaged inequality. Nor does it refute the unordered rainbow-path conjecture or the Caccetta–Häggkvist conjecture. We make no claim that six vertices is the minimum possible order.

The neighborhood table and the blow-up argument give a self-contained proof. The accompanying standard-library Python verifier independently enumerates increasing walks and multiplies integer adjacency matrices. It checks the table, all 720 vertex relabelings, and blow-ups for $1 \leq m \leq 12$; the argument above, not these finite tests, proves the result for every m. A mixed-integer optimization search located the table; neither solver optimality nor numerical infeasibility is used in the proof.

This is an unrefereed note. A bounded literature search did not locate this counterexample, but does not certify priority. AI-assisted tools supported the search, calculations, proof development, and manuscript preparation; the author is responsible for the claims and final text.

References

[1] Blair Dowling Sullivan, *A Summary of Problems and Results related to the Caccetta–Häggkvist Conjecture*, 2006, AIM 2006–13, arXiv:math/0605646. See the definitions on pp. 1–2 and Conjectures 6.1 and 6.10 on pp. 6–7.