

An Exact Logarithmic-State Algorithm for Recursive Tight-Cycle Constructions

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Abstract

The classical recursive construction for tight-cycle-free three-uniform hypergraphs has optimal edge count $R(n) = \max_{0 \leq m < n} \{m \binom{n-m}{2} + R(m)\}$. We give an exact algorithm for a single input n that visits only $O(\log(n+2))$ states. Completing a cubic potential to a square shows that every maximizing split is one of the two integers surrounding $((2n+1)\sqrt{3} - (2n+3))/4$. The descendants of these two choices occupy at most three consecutive integers at each depth, and their sizes decrease by at least a factor of two. The same identity gives an explicit linear error bound and the second-order expansion $R(n) = (\sqrt{3}/3 - 1/2)n^3 + (\sqrt{3} - 2)n^2/2 + O(n)$. Integer square roots suffice for exact evaluation and reconstruction of an optimal block sequence. Nearest-integer rounding alone already fails at $n = 106$. These results optimize the established recursive family; they do not resolve the single-forbidden-tight-five-cycle Turán density conjecture.

1 The finite recursive optimization

Mubayi and Rödl [1, Construction 1] partition a vertex set into A and B , include all triples meeting A in exactly two vertices, and repeat the construction inside B . This gives the familiar lower bound $2\sqrt{3} - 3$ for the Turán density of the tight five-cycle

$$C_5^3 = \{123, 234, 345, 451, 512\}.$$

The conjecture that this is the full density appears in the AIM workshop list [2, Section 2.4, Conjecture 7], and is recorded as AIM-COMBINATORICS-0153 in UnsolvedMath v1.6.0 [7].

Let $R(n)$ be the maximum edge count in this recursive family. Its finite recurrence is established in [3, Section 4, equations (3)–(4)] and [4, Section 1]:

$$R(0) = 0, \quad R(n) = \max_{0 \leq m < n} \{w(n, m) + R(m)\}, \quad w(n, m) = m \binom{n-m}{2}. \quad (1)$$

In particular $R(1) = R(2) = 0$. The option $m = 0$ simply terminates the construction and does not change its maximum. Every decreasing sequence $n = n_0 > n_1 > \dots > n_t = 0$ specifies the successive block sizes $n_j - n_{j+1}$ and has edge count $\sum_{j < t} w(n_j, n_{j+1})$.

The construction, recurrence, and leading coefficient are not new. Our subject is the exact integer optimization in (1). Direct dynamic programming examines $\Theta(n^2)$ splits to compute all

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values up to n . We show that a single value and an optimal block sequence require only logarithmically many states. The argument also identifies the quadratic coefficient, with an explicit $O(n)$ remainder. No claim of absolute historical priority is made.

Throughout, put

$$\beta = \frac{\sqrt{3}-1}{2}, \quad \gamma = 1-\beta = \frac{3-\sqrt{3}}{2}, \quad \delta = \frac{\gamma}{2}, \quad \kappa = \frac{1}{\sqrt{3}}, \quad t_n = \beta n - \delta. \quad (2)$$

Thus β is the asymptotic proportion of the part in which recursion continues; some references instead name the complementary proportion.

2 A square potential and the quadratic term

Define the polynomial

$$U(n) = \left(\frac{\sqrt{3}}{3} - \frac{1}{2}\right)n^3 + \frac{\sqrt{3}-2}{2}n^2 + \left(\frac{\sqrt{3}}{4} - \frac{3}{8}\right)n. \quad (3)$$

Theorem 1. *For real n, m , with $w(n, m) = m(n-m)(n-m-1)/2$ interpreted polynomially,*

$$U(n) - w(n, m) - U(m) = \kappa(n-m)(m-t_n)^2. \quad (4)$$

For every integer $n \geq 0$,

$$U(n) - \frac{n}{4\sqrt{3}} \leq R(n) \leq U(n). \quad (5)$$

In particular,

$$R(n) = \left(\frac{\sqrt{3}}{3} - \frac{1}{2}\right)n^3 + \frac{\sqrt{3}-2}{2}n^2 + O(n). \quad (6)$$

Proof. Write $a = \sqrt{3}/3 - 1/2$, $c = -\beta^2 = (\sqrt{3}-2)/2$, and $d = \sqrt{3}\beta^2/4 = \sqrt{3}/4 - 3/8$. The cubic terms satisfy

$$a(n^3 - m^3) - \frac{1}{2}m(n-m)^2 = \kappa(n-m)(m-\beta n)^2.$$

The remaining quadratic terms are

$$c(n^2 - m^2) + \frac{1}{2}m(n-m) = \beta(n-m)(m-\beta n).$$

Since $2\kappa\delta = \beta$ and $\kappa\delta^2 = d$, adding $d(n-m)$ completes the square, proving (4).

Telescoping along every decreasing integer path gives the exact identity

$$U(n) - R(n) = \min_{n=n_0 > \dots > n_t=0} \kappa \sum_{j < t} (n_j - n_{j+1})(n_{j+1} - t_{n_j})^2. \quad (7)$$

All terms are nonnegative, giving the upper bound. For the lower bound, at each positive state v choose the integer nearest to t_v . It is admissible: $t_v \geq t_1 = (3\sqrt{3}-5)/4 > 0$ and

$$v - \frac{1}{2} - t_v = \gamma v - \frac{\beta}{2} > 0.$$

There is no rounding tie because t_v is irrational. Each squared deviation is less than $1/4$, and $\sum_{j < t} (n_j - n_{j+1}) = n$. This path therefore loses at most $\kappa n/4$ from $U(n)$. The case $n = 0$ is immediate, and (6) follows. $\square$

The bounds specify a linear remainder interval. They do not assert convergence of the remainder divided by n , or exact optimality of the nearest-integer path.

3 Two possible maximizing splits

Theorem 2. *For every $n \geq 1$, every maximizing split in (1) belongs to*

$$\{\lfloor t_n \rfloor, \lceil t_n \rceil\} \cap \{0, \dots, n-1\}. \quad (8)$$

Thus maximizing over these two candidates computes exactly $R(n)$.

Proof. For a maximizing m , set $q = n - m \geq 1$ and $x = m - t_n$. Equations (4) and (5) imply

$$\kappa qx^2 \leq \kappa qx^2 + U(m) - R(m) = U(n) - R(n) \leq \kappa n/4.$$

Hence $qx^2 \leq n/4$, while $q = \gamma n + \delta - x$. If $x \leq -1$, then $q > \gamma n$ and $qx^2 > \gamma n > n/4$, a contradiction.

Suppose instead that $x \geq 1$. Substituting $n = (q + x - \delta)/\gamma$ yields

$$\begin{aligned} 4\gamma qx^2 - q - x + \delta &\geq 4\gamma x^2 - 1 - x + \delta \\ &\geq 4\gamma - 2 + \delta = \frac{9}{2}\gamma - 2 > 0. \end{aligned}$$

The first inequality uses $q \geq 1$ and $4\gamma x^2 - 1 > 0$. The second follows since $4\gamma x^2 - x - 1 + \delta$ is increasing on $x \geq 1$. This contradicts $qx^2 \leq n/4$ again. Therefore $|m - t_n| < 1$, proving (8). $\square$

Both candidates must in general be tested. At $n = 106$,

$$t_{106} = \frac{213\sqrt{3} - 215}{4} \in (38, 38.5).$$

The nearest integer is 38, but the two continuation scores are

$$w(106, 38) + R(38) = 90\,614, \quad w(106, 39) + R(39) = 90\,615.$$

Thus nearest-integer rounding can fail even when the continuation inside the chosen part is optimized.

4 An exact compressed algorithm

Theorem 3. *For a given integer $n \geq 0$, $R(n)$ and one optimal sequence of block sizes can be computed with $O(\log(n+2))$ states and $O(\log(n+2))$ integer arithmetic and integer square-root operations. All intermediate integers have $O(\log(n+2))$ bits. In particular, the algorithm has polynomial bit complexity in the binary input length.*

Proof. The values at 0, 1, 2 are known. Start with the layer $\{n\}$ if $n \geq 3$. At each depth replace every nonterminal state v by its two candidates in (8); discard repeated and terminal states. If the current layer has diameter at most two, its centers have diameter at most $2\beta < 1$. Their floors differ by at most one, so all children lie in at most three consecutive integers. Induction from the singleton initial layer bounds every layer's size by three.

Every child of $v \geq 3$ is at most $\lfloor v/2 \rfloor$. For even v , use $t_v < v/2$. For odd $v = 2k + 1$, $k \geq 1$, use

$$t_v - k = (2\beta - 1)k + \frac{3\beta - 1}{2} \leq \frac{7\beta - 3}{2} < 0,$$

where $\beta < 3/7$. Thus there are $O(\log(n+2))$ layers.

Evaluate them in reverse order using (1) restricted to (8). Each state has at most two children, whose values have already been computed in the next layer or are terminal. Store a maximizing child at every state; following these choices recovers an optimal block sequence of logarithmic length. Repeated states in different layers may be recomputed without changing the stated bounds.

All split computations can be exact:

$$k_v := \lfloor t_v \rfloor = \left\lfloor \frac{\text{isqrt}(3(2v+1)^2) - (2v+3)}{4} \right\rfloor, \quad (9)$$

where $\text{isqrt}(z) = \lfloor \sqrt{z} \rfloor$. Indeed, taking an inner floor, subtracting an integer, and dividing by the positive integer 4 does not alter the outer floor. The candidates are $k_v, k_v + 1$. States, rewards, potential-free continuation values, and integer square-root arguments all have $O(\log(n+2))$ bits. Standard integer arithmetic and square-root algorithms give the asserted polynomial bit complexity. $\square$

The operation count does not treat arbitrarily large integers as constant-cost machine words. Nor does it include listing the $\Theta(n^3)$ individual edges: the output is the integer value and its compressed block description.

5 Consequences and the scope of the conjecture

The finite extremal interpretation of $R(n)$ as the maximum number of induced out-directed three-vertex stars in an oriented graph is already known. In the colored-complete-graph formulation, these stars are called cherries; see [3, Theorem 2.5 and its proof], which credits Falgas-Ravry–Vaughan [5] and Huang [6]. The recursive orientation, with all arcs from B to A and no arcs inside A , attains (1). Theorem 3 therefore also computes this known finite extremal value; it is not a new symmetrization theorem.

Bodnár, León, Liu and Pikhurko [4, Theorem 1.3 and equation (3)] establish

$$\text{ex}(n, \{C_4^3, C_5^3\}) = R(n) + O(1). \quad (10)$$

Combining their structural theorem with Theorem 1 gives the following consequence.

Corollary 4. *For the pair of forbidden tight cycles,*

$$\text{ex}(n, \{C_4^3, C_5^3\}) = \left(\frac{\sqrt{3}}{3} - \frac{1}{2} \right) n^3 + \frac{\sqrt{3}-2}{2} n^2 + O(n).$$

The single forbidden C_5^3 problem is different. The same authors [4, p. 3] show that one may add linear three-graphs inside successive first parts without creating a tight five-cycle, improving $R(n)$ by a quadratic-order number of edges. Consequently $R(n)$ is not the exact single-cycle Turán number. An upper bound for arbitrary C_5^3 -free hypergraphs remains outside the results here. In particular, the density conjecture in the introduction is not proved.

6 Exact reproducibility

The supplementary standard-library Python implementation uses integer square roots and arithmetic in $\mathbb{Q}(\sqrt{3})$. It checks the polynomial identity coefficient by coefficient and compares the sparse

algorithm with exhaustive dynamic programming for every $0 \leq n \leq 3000$, including every maximizing split. This examines 4 501 500 split scores. It also reconstructs optimal paths for inputs as large as $2^{4096} + 12345$. The 30 039 conditions pass identically with and without Python optimization. These computations test the implementation; the universal statements follow from the proofs above, not from finite sampling or a formal proof assistant.

This manuscript was developed with AI assistance in derivation, source comparison, exposition and implementation. It is an unrefereed preprint.

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