

Critical Groups Detect Homology Actions of Simple Graphs

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Abstract

For every finite connected simple graph, we prove that the natural actions of its automorphism group on the finite critical group and on real first homology have the same kernel. The proof uses the cut–flow presentation and the absence of integral flows of squared norm at most two. A separate coordinate argument handles automorphisms reversing an edge. Combining the kernel equality with known homological faithfulness gives a faithful critical-group action for every connected simple graph of minimum degree at least two other than a cycle. In particular, no semiregularity assumption is needed for simple three-connected graphs. Parallel edges give explicit counterexamples to the kernel equality.

1 The kernel equality

Throughout the main theorem, G is a finite connected simple undirected graph. Its critical group is the finite abelian group

$$\text{Jac}(G) = \text{Div}^0(G)/L\mathbb{Z}^{V(G)},$$

where L is the positive graph Laplacian and $\text{Div}^0(G)$ is the group of integer vertex vectors with coordinate sum zero. We distinguish this finite group from the real Jacobian torus. Vertex permutation gives a natural representation

$$\rho_J : \text{Aut}(G) \longrightarrow \text{Aut}(\text{Jac}(G)).$$

The signed action on oriented edges restricts to the usual homology representation

$$\rho_H : \text{Aut}(G) \longrightarrow \text{GL}(H_1(G, \mathbb{R})).$$

Theorem 1. *For every finite connected simple graph,*

$$\ker \rho_J = \ker \rho_H.$$

The equality includes graphs with bridges and trees. Faithfulness itself requires further hypotheses.

Corollary 2. *Let G be a finite connected simple graph.*

1. *If G is bridgeless and has cycle rank at least two, then ρ_J is faithful.*
2. *More generally, if G has minimum degree at least two and is not a cycle, then ρ_J is faithful.*
3. *For a cycle C_n , $n \geq 3$, the kernel of ρ_J is exactly its rotation subgroup.*

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Proof. Baker and Norine prove homological faithfulness under the hypotheses of (1) [3, Proposition 4.23]. Estélyi et al. prove it under the hypotheses of (2) [4, Proposition 5]. Apply Theorem 1. On the one-dimensional homology of a cycle, rotations act as $+1$ and reflections as -1 , which proves (3). $\square$

Estélyi et al. [5, Theorem 4.3] prove that a subgroup acting semiregularly on vertices and darts of a connected simple three-edge-connected graph acts faithfully on its Jacobian. Theorem 1 removes semiregularity and leads to the broader minimum-degree condition in Corollary 2. In particular, every simple three-connected graph has a faithful natural critical-group action. This answers the simple-graph version of [5, Problem 1] negatively. Their paper also discusses multigraphs; our simplicity hypothesis must not be dropped.

Levine's item 41 in the AIM chip-firing workshop list [6] suggests studying the critical group together with a harmonic group action. The theorem treats one precise question within that programme. It concerns arbitrary automorphism actions, and hence their harmonic subgroups, but does not claim to exhaust that open-ended research direction. The cut-flow description and the homological faithfulness results are established inputs; the argument here compares the two kernels directly.

2 The cut-flow presentation

Choose one orientation for each edge and let

$$C = \mathbb{R}^{E(G)}, \quad B : C \longrightarrow \mathbb{R}^{V(G)}$$

be the signed incidence map, with Be equal to the terminal vertex minus the initial vertex of the oriented edge e . Give C its orthonormal edge basis. Set

$$H = \ker B, \quad \Lambda = H \cap \mathbb{Z}^{E(G)}, \quad N = \operatorname{im}(B^\top) \cap \mathbb{Z}^{E(G)},$$

and write $\pi : C \rightarrow H$ for orthogonal projection. Here $H = H_1(G, \mathbb{R})$, Λ is the integral flow lattice, and N is the integral cut lattice.

We recall the elementary presentation underlying the classical lattice description of the critical group [1, 2].

Lemma 3. *There are natural automorphism-equivariant isomorphisms*

$$\operatorname{Jac}(G) \cong \mathbb{Z}^{E(G)} / (\Lambda + N) \cong \pi(\mathbb{Z}^{E(G)}) / \Lambda.$$

Moreover, $N = B^\top \mathbb{Z}^{V(G)}$.

Proof. If $B^\top f$ is an integer edge vector for a real vertex potential f , subtract $f(r)$ at a fixed root r . Differences along edges are integers, so differences along a path from r show that every coordinate of the resulting potential is an integer. This proves the assertion about N .

Connectedness gives $B\mathbb{Z}^{E(G)} = \operatorname{Div}^0(G)$, since vertex differences are boundaries of paths. The integer kernel of B is Λ and $L = BB^\top$. The inverse image under B of $L\mathbb{Z}^{V(G)}$ is therefore $\Lambda + B^\top \mathbb{Z}^{V(G)}$, which proves the first isomorphism.

Orthogonal projection annihilates N and is the identity on Λ . Conversely, if an integer edge vector x satisfies $\pi x \in \Lambda$, then $x - \pi x$ is both integral and in $H^\perp = \operatorname{im} B^\top$, so it belongs to N . This proves the second isomorphism. Each construction commutes with graph automorphisms. $\square$

An automorphism σ induces an orthogonal signed permutation S of the edge basis. Since S preserves H , it commutes with π . Thus, if $\sigma \in \ker \rho_J$, Lemma 3 gives

$$\pi(Se - e) \in \Lambda \quad (e \in E(G)). \quad (1)$$

It is this integrality, together with a short-vector bound, that forces S to fix the cycle space pointwise.

3 Proof of the main theorem

Lemma 4. *For a simple graph, every nonzero vector of Λ has squared norm at least three. For every edge basis vector e ,*

$$0 \leq \langle \pi e, e \rangle = \|\pi e\|^2 < 1.$$

Proof. The support of a nonzero integral circulation has no vertex of degree one: at such a vertex the zero-boundary condition would force the unique supported coefficient to vanish. A finite nonempty support therefore contains a cycle, and a cycle in a simple graph has at least three edges. The nonzero integer coefficients on those edges contribute at least three to the squared norm.

The equality $\langle \pi e, e \rangle = \|\pi e\|^2$ follows from orthogonal projection. Its value is at most $\|e\|^2 = 1$. Equality would imply $e \in H$, whereas $Be \neq 0$ because an edge has distinct endpoints. $\square$

The first statement is also the special case of the known equality between the minimum squared norm of the flow lattice and the girth [1, Proposition 1(ii)]. We included the direct argument to make the norm threshold explicit.

Proof of Theorem 1. Let $\sigma \in \ker \rho_J$. For an edge basis vector e , put $z = \pi(Se - e) \in \Lambda$.

If $Se = e$, then $z = 0$. If $Se = \pm f$ for a different edge f , then

$$\|z\|^2 \leq \|Se - e\|^2 = 2.$$

Lemma 4 again gives $z = 0$.

There remains the case $Se = -e$, for which the preceding estimate would only give four. Put $h = \pi e$ and $a = \|h\|^2$. Then $z = -2h$ and its e -coordinate is

$$z_e = -2\langle h, e \rangle = -2a.$$

By Lemma 4, $0 \leq a < 1$. If $a = 0$, then $z = 0$. If $a > 0$, integrality of z_e forces $z_e = -1$ and $a = 1/2$. But this gives $\|z\|^2 = 4a = 2$, again impossible for a nonzero integral flow. Therefore $z = 0$ in every case.

The vectors πe span H , and

$$S\pi e = \pi Se = \pi e.$$

Thus S fixes H pointwise and $\sigma \in \ker \rho_H$.

Conversely, suppose S fixes H pointwise. Then $\pi(Se - e) = 0$ for every edge e . The integer vector $Se - e$ belongs to N , so S acts identically on $\mathbb{Z}^{E(G)}/(\Lambda + N)$. Lemma 3 gives $\sigma \in \ker \rho_J$. $\square$

4 Exceptions and exact checks

The failure for multigraphs is immediate and structural. Give the three edges of a two-vertex banana graph the same orientation. A transposition of two edges fixes every vertex divisor, so it fixes $\text{Jac}(G) \cong \mathbb{Z}/3\mathbb{Z}$. It does not fix homology: it sends the integral flow $(1, -1, 0)$ to $(-1, 1, 0)$. The squared norm of that flow is two, precisely the threshold excluded by simplicity.

Bridges do not invalidate Theorem 1, but they can allow a nontrivial common kernel. For example, attach two leaves to one vertex of a triangle. Swapping the leaves fixes the triangle's cycle space and therefore fixes the critical group. Removing the minimum-degree condition from Corollary 2(2) would consequently be false. The cycle exception is also necessary.

As a diagnostic independent of the proof, we tested the connected simple graphs on two through seven vertices in the standard graph atlas. One implementation computes principal-divisor membership by rational inversion of a reduced Laplacian and enumerates automorphisms with VF2. A second enumerates degree-compatible vertex permutations directly, tests the Jacobian action using integer cofactors and divisibility, and compares it with the signed action on a

fundamental cycle basis. The independent implementation checked 995 connected graph types and 13,286 automorphisms, with no kernel discrepancy. On the 577 bridgeless graph types it also reproduced every kernel from the first implementation. Both ordinary and optimized Python runs passed. These finite tests are supplementary; they do not replace the uniform proof.

The accompanying source package includes the two exact programs and their result records. NetworkX 3.5 supplies the atlas and, only in the first implementation, the VF2 enumeration. No numerical tolerance is used.

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