

Orientation Dependence and Exact Groups for Ridge-Routing Sandpiles

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Abstract

A fitting orientation of a simplicial rooted forest specifies an unsigned chip-firing rule on its nonroot ridges. We show that the resulting sandpile group can depend on that orientation: in every dimension $d \geq 2$, a six-facet rooted forest with relative boundary determinant one admits cyclic groups of orders $d^6 - d^3 - d$ and $d^6 - 2d$. In contrast, if each ridge belongs to at most two facets, the abstract group is independent of the fitting orientation. We give its cyclic decomposition and exact formulas for recurrent states, the identity, single-chip avalanches and individual addition periods. Even in this thin case, labeled addition periods can change. The proofs use an integral cycle-and-chain splitting and standard directed-sandpile theory. These results analyze a specified routing model; they do not settle the general AIM request for a higher-dimensional chip-firing model or establish a canonical torsor.

1 The model and the main distinction

Question 22 of the 2013 AIM workshop list asks for a chip-firing model for directed rooted forests in higher dimension [1]. Bernardi and Klivans define fitting orientations using incidences between facets and nonroot ridges [2, Definition 26]. A frozen research attempt accompanying the dataset record AIM-COMBINATORICS-0112 proposes the following unsigned routing rule and proves its absorption on simplicial forests [6]. We take that rule as a credited starting point, not as a new construction.

Let F be a finite collection of d -simplices, with $d \geq 2$, and let E be all their $(d-1)$ -faces, called ridges. Choose roots $R \subseteq E$, and set $V = E \setminus R$. A *fitting orientation* is a bijection $\phi : V \rightarrow F$ such that $v \subset \phi(v)$. Firing v removes d chips there and sends one to each other ridge of $\phi(v)$. Chips reaching R disappear. Write D_ϕ for the digraph on V with an arc $v \rightarrow w$ whenever $w \neq v$ is another ridge of $\phi(v)$. Its source-row adjacency matrix is A_ϕ , so the column toppling matrix and routing group are

$$T_\phi = dI - A_\phi^\top, \quad K_\phi = \mathbb{Z}^V / T_\phi \mathbb{Z}^V.$$

All missing outgoing arcs lead to one sink. The nonroot digraph is simple and loopless. Let B be the signed simplicial boundary matrix with rows V and columns F . By [2, Lemma 14], the pair (F, R) is a rooted forest precisely when B is square and nonsingular.

Call F *thin* if every ridge is contained in at most two of its facets. Our central distinction is the following.

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Theorem 1.1. *For a fixed thin simplicial rooted forest (F, R) , K_ϕ is independent of the fitting orientation as an abstract group. It need not be independent as a group with labeled ridge-addition generators. For every $d \geq 2$, there is a d -dimensional rooted forest with six facets on $d+3$ vertices, $\det B = 1$, and maximum ridge incidence three, with two fitting orientations whose groups are*

$$\mathbb{Z}/(d^6 - d^3 - d)\mathbb{Z} \quad \text{and} \quad \mathbb{Z}/(d^6 - 2d)\mathbb{Z}.$$

Thus incidence two is a sufficient uniform bound for abstract invariance, whereas incidence three is not. Sections 5 and 6 prove the theorem. The matrix T_ϕ is not the signed cellular Laplacian, and its cokernel is not relative homology torsion. For one simplex with one nonroot ridge, for example, $K_\phi = \mathbb{Z}/d\mathbb{Z}$ while $|\det B| = 1$.

The linear-algebra background is the cycle-and-chain decomposition of functional digraph matrices, including the difference-vector construction of Cardon and Tuckfield [3, Theorem 6 and Lemma 10]. We give an integral version needed for cokernels, without claiming a new Jordan-form theorem. Standard global-sink sandpile results are taken from Holroyd et al. [4]. The comparisons between fitting orientations, discrete gradients, orientability and branching in [2, 5] are also relevant. Novelty of the exact synthesis below remains undetermined after a bounded comparison with primary literature; no absolute-priority claim is made.

2 Absorption and tight sets

First consider a finite indexed family F of nonempty subsets of a finite set E , admitting a matching to distinct incident elements of E . Matched elements are nonroots, and each routes to all other elements of its matched member of F . Unmatched elements are roots. For $C \subseteq F$, let $N(C)$ be the union of its supports.

Proposition 2.1. *Every nonroot reaches a root for one such matching if and only if this holds for every matching, if and only if*

$$|N(C)| > |C| \quad (\emptyset \neq C \subseteq F).$$

The nonabsorbing set is matching independent: it is the union of $N(C)$ over all tight collections with $|N(C)| = |C|$.

Proof. Fix a matching, and let U be all nonroots from which no root is reachable. Every element of the facet assigned to an element of U belongs to U . The collection C of those facets therefore satisfies $N(C) = U$ and $|C| = |U|$. Conversely, if $|N(C)| = |C|$, every matching uses every element of $N(C)$ to match the facets of C . Hence $N(C)$ contains no root and is closed under all routing arrows. It is nonabsorbing for every matching. These two implications prove both the union formula and the equivalences. $\square$

This is the usual Hall-tight-set mechanism, not a new matching theorem. For a simplicial forest, the boundary columns indexed by a nonempty C are independent. Thus $|N(C)| \geq |C|$. Equality would make their restriction M to $N(C)$ square and invertible. The chain identity gives $\partial_{d-1}|_{N(C)}M = 0$, forcing every ridge in $N(C)$ to have zero boundary, which is impossible for a positive-dimensional simplex. Consequently strict Hall holds. This recovers the upstream absorption argument. In dimension one the same argument uses augmented boundary. There is also a simpler incidence criterion: if every ridge belongs to at most q facets, then

$$(d+1)|C| \leq q|N(C)|.$$

In particular $q \leq d$ suffices for strict Hall without a forest assumption.

For a global-sink digraph, legal stabilization terminates and is independent of firing order. Stable configurations satisfy $0 \leq c_v \leq d - 1$. Recurrent configurations form a group under stabilized addition, have cardinality $\det T$, and represent the lattice classes of $\mathbb{Z}^V / T\mathbb{Z}^V$ uniquely; see [4, Lemmas 2.2, 2.4, 2.8, 2.13, 2.15 and Corollary 2.16]. We transpose that reference's row convention to use column chip vectors throughout.

3 The thin graph and its integral group

In a thin complex, each nonroot has indegree at most one: of the at most two facets containing w , one is already $\phi(w)$ and cannot route into w . Its outdegree is at most d . Reversing all arrows yields a partial functional digraph. Each weak component of D_ϕ is therefore an outward rooted tree or a directed cycle with outward trees attached. No noncycle vertex has a path into a cycle.

These properties alone imply absorption when $d \geq 2$. A nonempty set S closed away from the sink would have $d|S|$ internal outgoing arcs and at most $|S|$ internal incoming arcs, a contradiction. Equivalently, the column sums of A are at most one, so $\rho(A) \leq 1 < d$. All results of this section and the next apply to any finite loopless digraph with indegree at most one and outdegree at most d , using a sink for the missing chips.

Delete all cycle vertices. In the remaining outward forest, let h_j count the vertices of height exactly j , where height is the greatest distance to a descendant leaf. Put $h_j = 0$ for large j . Let $\ell_1, \dots, \ell_c$ be the deleted cycle lengths and f the number of noncycle vertices.

Theorem 3.1. *There is an abstract isomorphism*

$$K \cong \bigoplus_{k \geq 1} (\mathbb{Z}/d^k \mathbb{Z})^{h_{k-1} - h_k} \oplus \bigoplus_{i=1}^c \mathbb{Z}/(d^{\ell_i} - 1)\mathbb{Z}. \quad (1)$$

The displayed factors need not be in invariant-factor order. In particular,

$$\det T = d^f \prod_{i=1}^c (d^{\ell_i} - 1).$$

Proof. Regard the source-row matrix A as acting on the standard column basis. It sends a child's basis vector to its parent's vector; a root's vector goes to zero. On each cycle it is a permutation P . If v lies at depth t from its cycle attachment a , replace its vector by $b_v = e_v - P^{-t}e_a$. Leave cycle vectors fixed. This is a unimodular change of basis, since each noncycle column retains its diagonal one and only cycle coordinates have been subtracted. Now Ab_v is the new parent vector, or zero if $t = 1$. Thus the permutation module splits integrally from the forest parent map N .

In each rooted tree choose a longest root-to-leaf path $v_0, \dots, v_h$. For an off-path vertex x at depth s , replace e_x by $e_x - e_{v_s}$ and leave the path vectors fixed. The index exists by maximality of the path. The first vertex of an off-path branch maps to zero in the new basis, and every subsequent vertex maps to its new parent vector. This unimodular transformation splits off a nilpotent path block. Repeat on the branches. This gives an integral chain decomposition by the same elementary difference-vector mechanism as [3].

The rank of N^j counts vertices with a descendant at distance j . Consequently $\text{rank } N^j - \text{rank } N^{j+1} = h_j$. The number of nilpotent blocks of length k is therefore $h_{k-1} - h_k$, necessarily nonnegative. Successive eliminations at off-diagonal unit entries show that $dI - J_k$ has cyclic cokernel of order d^k for a nilpotent path block J_k . The analogous elimination around a permutation cycle of length ℓ gives a cyclic group of order $d^\ell - 1$. Finally $T = (dI - A)^\top$ has the same Smith

form as $dI - A$. This proves the abstract group formula and its order. Transposition is used only here, not to identify labeled generators. $\square$

4 Recurrence, avalanches and labeled periods

Theorem 4.1. *A stable configuration is recurrent if and only if each directed cycle contains at least one positive chip count. The recurrent identity is zero on acyclic components and is $d - 1$ at every vertex of every component containing a cycle.*

Proof. A positive cycle cannot become all zero under additions and legal firings. If there is no cycle firing, positivity remains. Otherwise the last cycle firing leaves a chip at its successor, which does not fire later. Thus a stable state with an all-zero cycle is inaccessible from the maximal stable configuration and is not recurrent. There are exactly $d^f \prod_i (d^{\ell_i} - 1) = \det T$ stable states satisfying the asserted condition. The standard recurrent-state count therefore proves sufficiency.

Zero is recurrent on an acyclic component and represents the zero class. On a cyclic component every vertex has indegree one, so $T\mathbf{1} = (d - 1)\mathbf{1}$. This stable recurrent vector represents zero. Uniqueness of the recurrent representative proves the identity formula. $\square$

Theorem 4.2. *Start from a stable state c and add one chip at v . Every vertex fires at most once. If $c_v < d - 1$ there is no firing. Otherwise the fired set consists exactly of the vertices reachable from v in the subgraph induced by $\{w : c_w = d - 1\}$.*

Proof. Before the first second firing, each vertex has fired at most once. The vertex about to fire again has at most one predecessor, so its initial chips, possible external chip and possible incoming chip total at most $d + 1 < 2d$. Two firings would require $2d$ chips, a contradiction. Hence a vertex other than v fires exactly when it was initially full and its predecessor fires. The initially full v is the only possible starting point. These implications give the stated reachability rule. $\square$

For a noncycle vertex v , write $H(v)$ for its outward height. For a cycle vertex a , let H_a be the greatest depth in the trees attached to a , with $H_a = 0$ if there are none. For vertices v, a on a cycle C of length ℓ , write $t(v, a) \in \{0, \dots, \ell - 1\}$ for the forward cyclic distance and put

$$D_v = \max_{a \in C} (t(v, a) + H_a).$$

Theorem 4.3. *The period of addition at v on every recurrent state is*

$$\begin{cases} d^{H(v)+1}, & v \text{ is not on a cycle,} \\ (d^\ell - 1)d^{D_v+1-\ell}, & v \text{ is on a cycle of length } \ell. \end{cases}$$

Proof. Addition at v is translation by $[e_v]$. Its order is the least positive integer q making $qT^{-1}e_v$ integral. Since $\rho(A) < d$,

$$T^{-1}e_v = \sum_{k \geq 0} \frac{(A^\top)^k e_v}{d^{k+1}}.$$

If v is not on a cycle, each reachable descendant at distance δ has exactly one walk from v and entry $d^{-\delta-1}$. Their largest denominator is $d^{H(v)+1}$. If v is on a cycle, walks to a reachable w have lengths $\delta + k\ell$ for $k \geq 0$, where δ is the shortest distance. The corresponding entry is

$$\frac{d^{\ell-\delta-1}}{d^\ell - 1},$$

whose reduced denominator is $(d^\ell - 1)d^{\max(0, \delta+1-\ell)}$ because $\gcd(d, d^\ell - 1) = 1$. The largest shortest distance is $D_v \geq \ell - 1$. Taking the least common multiple proves the formula. It concerns recurrent states, not arbitrary transient configurations. $\square$

5 Changing the fitting orientation in a thin forest

Proposition 5.1. *Fix a thin rooted forest (F, R) . Let c be the number of unicyclic components of its nonroot dual graph. Then there are 2^c fitting orientations, $|\det B| = 2^c$, and all the abstract groups K_ϕ are isomorphic.*

Proof. The dual graph has one vertex for each facet. A nonroot ridge shared by two facets gives an ordinary edge; a nonroot ridge in one facet gives a half-edge. The boundary matrix has no zero row. Its connected blocks are rectangular diagonal blocks. Nonsingularity of B forces each block to have full row and column rank, hence to be square and invertible. A connected block with n vertices, e edges and h half-edges therefore has $e + h = n$ and $e \geq n - 1$. It is either a tree with one half-edge or a unicyclic graph with no half-edge.

In the tree case, matching is forced toward the half-edge. In the unicyclic case, stripping off-cycle leaves forces all fringe matches, leaving the two cycle directions. Reversing a cycle leaves the induced off-cycle ridge-routing forest unchanged, although it can change where that forest attaches to the cycle. Thus its height counts and the cycle lengths are unchanged. Theorem 3.1 proves group invariance, and the independent cycle choices give 2^c matchings.

Leaf elimination in B uses unit pivots. A tree block leaves determinant of absolute value one. A unicyclic block leaves a signed cycle matrix with two determinant terms, each of absolute value one. Its determinant is zero or has absolute value two; nonsingularity excludes zero. This proves the determinant formula. $\square$

If the facets can be coherently oriented along all shared ridges, a component with no half-edge has signed row sums zero after orienting the columns. Such a block is singular. Hence $c = 0$ in this case. This is consistent with the known orientability, collapse and gradient results discussed in [5], and is not a claim to replace them.

5.1 A labeled obstruction in the projective plane

Take the six-vertex triangulation with ordered facets

$$012, 013, 024, 035, 045, 125, 134, 145, 234, 235.$$

Choose roots 01, 02, 03, 04, 15 and order the nonroots as

$$05, 12, 13, 14, 23, 24, 25, 34, 35, 45.$$

Every ridge has incidence two, and the relative boundary determinant is -2 . Two fitting matchings, listing the assigned ridge in facet order, are

$$\begin{aligned} (12, 13, 24, 05, 45, 25, 34, 14, 23, 35), \\ (12, 13, 24, 35, 05, 25, 14, 45, 34, 23). \end{aligned}$$

Both routing graphs have a cycle of length six and a fringe forest with three height-zero vertices and one height-one vertex. Hence

$$K \cong (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/252\mathbb{Z}, \quad |K| = 1008.$$

For the ridge 05, Theorem 4.3 gives periods 63 and 126, respectively. There can consequently be no group isomorphism fixing every labeled addition generator. The reproducibility files contain the exact boundary, toppling and inverse-column calculations; no minimality assertion is made.

6 A six-facet example in every dimension

Start with facets $F = (012, 013, 034, 123, 124, 234)$ on $0, \dots, 4$. Take roots $R = (02, 04, 13, 14)$ and order the nonroots as $V = (01, 03, 12, 23, 24, 34)$. With increasing vertex orders the boundary matrix is

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad \det B = 1.$$

This is a rooted forest. Its maximum ridge incidence is three. Consider the two fitting matchings, again written in facet order,

$$\alpha = (01, 03, 34, 12, 24, 23), \quad \beta = (12, 01, 03, 23, 24, 34).$$

Number the nonroots $0, \dots, 5$ in their displayed order. The nonroot arrows are as follows; other outgoing chips go to roots.

source	0	1	2	3	4	5
α targets	2	0	3	4, 5	2	1
β targets	1	5	0	2	2	3, 4

Use threshold d temporarily on these two digraphs. In the α presentation set $g = e_5$. The relations $de_i = \sum_{i \rightarrow j} e_j$ give

$$(e_0, e_1, e_2, e_3, e_4, e_5) = (d^2, d, d^3, d^4, d^5 - 1, 1)g.$$

The remaining relation is $(d^6 - d^3 - d)g = 0$. Every elimination uses a coefficient 1 or -1 , so it is an equivalent cyclic presentation, not merely a map onto a cyclic quotient. For β , put $g = e_3$. The same unit eliminations give

$$(e_0, e_1, e_2, e_3, e_4, e_5) = (d^2, d^3, d, 1, d^5 - 1, d^4)g$$

and the sole relation $(d^6 - 2d)g = 0$. Both orders are positive for $d \geq 2$, and their difference is $d(d^2 - 1) > 0$. In dimension two they are 54 and 60.

To realize threshold d by actual simplices, take a disjoint vertex set S of size $d - 2$ and replace each triangle f by $f \cup S$. Designate $r \cup S$, for the six old nonroots r , as the new nonroots; all other ridges are roots. Order the new vertices after $0, \dots, 4$. On these nonroot rows the signed boundary matrix is still exactly B : removing an original vertex has its old sign, while removing a vertex of S produces a root. The matchings extend by adjoining S . Their nonroot arrows are unchanged and all extra chips go to new roots. Every newly introduced ridge is contained in one facet, so the maximum ridge incidence is still three. This constructs the claimed rooted forests on $d + 3$ vertices in every dimension $d \geq 2$, completing Theorem 1.1.

7 Verification and scope

The supplementary exact-arithmetic checks compare Smith forms and determinants on 460 graphs, inverse-column periods in 2391 cases, and 87140 single-chip stabilizations under two legal firing orders. Closed strongly connected components of the addition graph give 337 independent computational checks of recurrence over 16596 stable configurations; direct iteration checks 1291 recurrent addition periods. Here “independent” describes a distinct calculation, not an outside review. A second suite checks 4802 matching/absorption comparisons on 1770 small hypergraphs, 180 explicit unimodular similarity matrices, the two symbolic cyclic presentations, 24 simplicial orientation lifts through dimension nine, and all 1296 spanning root trees of the displayed projective-plane triangulation. These finite checks supplement the general proofs; they are not substitutes for them.

The orientation-dependent examples obstruct forgetting the fitting orientation in this unsigned model. They do not obstruct the existence of other higher-dimensional chip-firing models. No torsor on all ambient rooted forests, identification with cellular torsion, weighted-threshold extension, or canonical labeled-group invariance is asserted. In particular the broad AIM source record is not counted as resolved.

The author used AI assistance in literature discovery, mathematical reasoning, symbolic computation, code preparation and drafting. The arguments received a detailed originating-researcher self-audit; this preprint has not undergone independent peer review or proof-assistant formalization. The frozen upstream model and known matching, linear algebra and sandpile results are explicitly credited above.

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