B-Spline Laws and Nonreal Zeros for Weighted-Cycle Internal DLA

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Abstract

We give an exact finite-time distribution for single-source internal diffusion limited aggregation driven by an arbitrary positive nearest-neighbour chain on the integers. The probability of each occupied interval is a B-spline value on the harmonic-scale knots of the chain. Lifting a cycle until its penultimate occupation yields a last-site law and a divided-difference formula for arbitrary positive edge weights. For the four-cycle we characterize the full attainable probability region and every reversible realizing resistance vector up to scale. The shifted last-site polynomial has only negative real zeros for every positive nearest-neighbour chain on three or four vertices, but a five-cycle with resistances $(10, 10, 1, 1, 20)$ has a nonreal conjugate pair. Thus five is the smallest possible cycle size for this obstruction. The homogeneous Eulerian and q-Eulerian laws and the spline identities used here are classical. This elementary note makes no absolute priority claim and does not resolve higher-dimensional IDLA last-site problems.

1 The process and the scope of the result

Fix transition probabilities $0 < p_i < 1$, $i \in \mathbb{Z}$. In the sequential internal diffusion limited aggregation (IDLA) process, every explorer starts at 0, moves from i to $i+1$ with probability p_i and to $i-1$ otherwise, and settles at its first unoccupied site. Explorers are independent in this fixed environment. The first explorer occupies 0 immediately. Write A_m for the occupied set after m explorers. Nearest-neighbour motion implies that A_m is an interval of m sites containing 0.

On the cycle $C_n = \mathbb{Z}/n\mathbb{Z}$, $n \geq 3$, use the same rule and let L be the last occupied vertex. The source is occupied first, so $\mathbb{P}(L = 0) = 0$. Question 3 of the 2013 AIM chip-firing workshop notes [1] suggests studying last-site laws beyond the fair cycle, including higher-dimensional sets and circulants. Its comment (d) suggests one-dimensional edge weights as a possible source of q-analogues. The source is an open-ended programme, not a universal real-rootedness conjecture. We address the weighted nearest-neighbour branch only.

The fair Eulerian law is proved in Mittelstaedt [4, Theorem 1.1], whose Section 3 also gives the homogeneous biased, major-index q-Eulerian law. Her Question 4 considers site-dependent transition probabilities. Nadeau–Tewari [5, Section 5.1] identify normalized remixed Eulerian numbers with IDLA probabilities, include the single-pile specialization, and use a circular one-hole construction. Their equation (5.9), and [6, Remark 4.4], identify the relevant Carlitz polynomials. These homogeneous laws are prior art. Inhomogeneous IDLA is also an established model: Enriquez, Lucas and Simenhaus [3] study the Sinai environment and its annealed arcsine limit. We make no new scaling-limit claim.

Our observation is that the elementary interval recurrence matches the Cox–de Boor recurrence on nonuniform harmonic-scale knots. It gives an exact finite-time formula for every fixed positive

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environment. We then determine the four-cycle probability image and give a five-cycle obstruction to extending real-rootedness. The frozen research aid attached to AIM-COMBINATORICS-0093 in [7] already contains the resistance recurrence; we credit that starting point. The explicit spline identification, inverse fibers and obstruction are the additional claims developed here. All ingredients are classical, and folklore or differently indexed prior statements are not excluded by our bounded literature search.

2 Harmonic scale and the B-spline law

Choose $s_0 > 0$ and recursively define positive increments in both directions by

$$\frac{s_i}{s_{i-1}} = \frac{1-p_i}{p_i}, \qquad t_0 = 0, \qquad t_{i+1} - t_i = s_i. \tag{1}$$

The knots t_i are strictly increasing. They need not diverge at either infinite end: only finite portions of the sequence will be used. For degree $d \geq 0$, define the classical B-splines by

$$\begin{aligned} N_{i,0}(x) &= \mathbf{1}_{[t_i,t_{i+1})}(x), \\ N_{i,d}(x) &= \frac{x-t_i}{t_{i+d}-t_i} N_{i,d-1}(x) + \frac{t_{i+d+1}-x}{t_{i+d+1}-t_{i+1}} N_{i+1,d-1}(x), \quad d \geq 1. \end{aligned} \tag{2}$$

All denominators are positive. This is the Cox–de Boor recurrence; our degree d is order $d+1$ in [2, (2.4)]. A degree-d spline is supported on $[t_i, t_{i+d+1}]$. For positive degree it vanishes at both endpoints.

Lemma 1. *Let $\ell < 0 < r$ be integers. An explorer started at 0 first hits r rather than ℓ with probability*

$$\frac{-t_\ell}{t_r - t_\ell}. \tag{3}$$

Proof. Equation (1) gives $p_i(t_{i+1}-t_i) = (1-p_i)(t_i - t_{i-1})$. Thus $h_i = (t_i - t_\ell)/(t_r - t_\ell)$ is harmonic for $\ell < i < r$, with $h_\ell = 0$ and $h_r = 1$. Exit is almost sure: from each interior vertex a monotone path to a boundary has at most $r - \ell$ steps and positive probability. The minimum of these finitely many path probabilities is positive, giving a uniform chance of exit in each block of $r - \ell$ steps. The finite Dirichlet problem has a unique solution, for instance by the maximum principle. Its value at 0 is therefore the hitting probability (3). □

Theorem 2. *For $m \geq 1$ and integers $\ell \leq 0 \leq r$ with $r - \ell + 1 = m$,*

$$\mathbb{P}(A_m = [\ell, r]) = N_{\ell-1,m}(0). \tag{4}$$

In particular, the degree is m, and the knots run from $t_{\ell-1}$ through t_{r+1}.

Proof. For $a, b \geq 0$, put $P_{a,b} = \mathbb{P}(A_{a+b+1} = [-b, a])$. Conditioning on the last added endpoint and applying Lemma 1 gives

$$P_{a,b} = \frac{-t_{-b-1}}{t_a - t_{-b-1}} P_{a-1,b} + \frac{t_{a+1}}{t_{a+1} - t_{-b}} P_{a,b-1}. \tag{5}$$

A term with a negative index is omitted, and $P_{0,0} = 1$. The coefficients come from different predecessor intervals; they need not sum to one.

Set $i = -b-1$ and $d = a+b+1$ in (2), evaluated at $x = 0$. The first lower-degree spline corresponds to $(a-1, b)$ and the second to $(a, b-1)$, with exactly the coefficients in (5). If exactly one of a, b is zero, the absent predecessor's spline has positive degree and vanishes at its support endpoint. For $a = b = 0$, the degree-one hat on $t_{-1} < t_0 = 0 < t_1$ has value one at 0. Induction on $a + b$ proves (4). □

The finite-interval proof needs no recurrence of the walk on $\mathbb{Z}$, stationarity of the environment, or uniform ellipticity over all sites. Common positive rescaling of all the s_i leaves every displayed probability unchanged.

3 Cycles and a divided-difference formula

Corollary 3. *Let $0 < p_i < 1$ be n-periodic, and construct the scale on its lift to $\mathbb{Z}$. For the corresponding IDLA process on C_n,*

$$\mathbb{P}(L = j) = N_{j-n,n-1}(0), \qquad 1 \le j \le n-1. \tag{6}$$

This probability is independent of the outgoing transition probability p_j at the candidate last site.

Proof. Before each expansion needed to reach $n-1$ occupied vertices, the aggregate is an arc of at most $n-2$ vertices. Its two unoccupied boundary vertices are distinct. Lift the occupied arc to its interval containing the source lift 0. An explorer cannot wrap through an unoccupied vertex without first settling, so the lifted process is exactly the line process up to this penultimate state. The arc missing j lifts to $[j-n+1, j-1]$, of size $n-1$. Theorem 2 gives (6). The final explorer reaches the sole missing vertex almost surely. Before this occupation no explorer departs from that vertex, proving the independence assertion. □

For distinct knots the classical divided-difference representation is

$$N_{i,d}(x) = (t_{i+d+1} - t_i) \sum_{k=i}^{i+d+1} \frac{(t_k - x)_+^d}{\prod\limits_{\substack{h=i\\h\neq k}}^{i+d+1} (t_k - t_h)}, \qquad d \ge 1. \tag{7}$$

Here $v_+ = \max(v, 0)$. Formula (7) is [2, (5.8)]; the explicit denominators follow by taking the leading coefficient of the Lagrange interpolant to the truncated power data. Therefore

$$\mathbb{P}(L = j) = (t_j - t_{j-n}) \sum_{k=j-n}^{j} \frac{(t_k)_+^{n-1}}{\prod\limits_{\substack{h=j-n\\h\neq k}}^{j} (t_k - t_h)}. \tag{8}$$

This is an algebraic closed form, not a floating-point stability assertion. The nonnegative recurrence (5) computes the entire cycle law in $O(n^2)$ arithmetic operations. This count is not a bit-complexity bound for rational computation.

Corollary 4. *Give edge $\{i, i+1\}$ of C_n positive conductance c_i, and write $r_i = 1/c_i$ for its resistance. In (8) one may take*

$$t_{k+1} - t_k = r_{k \bmod n}, \qquad t_0 = 0, \qquad R = \sum_{i=0}^{n-1} r_i. \tag{9}$$

Then $t_{k+n} = t_k + R$, and the prefactor $t_j - t_{j-n}$ is R.

Proof. The clockwise probability at i is

$$p_i = \frac{c_i}{c_{i-1} + c_i} = \frac{r_{i-1}}{r_{i-1} + r_i}.$$

Consequently $(1 - p_i)/p_i = r_i/r_{i-1}$ and the increments in (9) satisfy (1). □

Remark 5. For a general directed periodic chain, put $\Lambda = \prod_{i=0}^{n-1}(1-p_i)/p_i$. Its increments instead obey $s_{i+n} = \Lambda s_i$, and $t_{i+n} = \Lambda t_i + t_n$. Additive periodicity must not be imposed when $\Lambda \neq 1$. Corollary 3 remains valid.

For $p_i = 1/2$ one may take $t_i = i$, recovering the known law

$$\mathbb{P}(L = j) = \frac{\mathrm{A}(n-1, j-1)}{(n-1)!},$$

where $\mathrm{A}(m,k)$ counts permutations of m with k descents. The cardinal-spline identity is recorded in [8, Theorem 1.1(i)]; it is not new here. For homogeneous bias, $\rho = (1-p)/p$ and $[k]_\rho = 1 + \rho + \cdots + \rho^{k-1}$ turn (5) into

$$P_{a,b} = \frac{[b+1]_\rho}{[a+b+1]_\rho} P_{a-1,b} + \frac{\rho^b [a+1]_\rho}{[a+b+1]_\rho} P_{a,b-1}.$$

Multiplication by $[a+b+1]_\rho!$ gives the classical descent-major-index recurrence. This is a normalization check on the known q-Eulerian specialization [4, 5, 6].

4 The exact four-cycle probability image

Theorem 6. *As positive conductances on C_4 vary, the possible laws $(x,y,z) = (\mathbb{P}(L=1), \mathbb{P}(L=2), \mathbb{P}(L=3))$ are exactly*

$$x, y, z > 0, \qquad x + y + z = 1, \qquad \sqrt{x} + \sqrt{z} < 1. \tag{10}$$

For each such law, every realizing resistance vector, modulo common positive scaling, is uniquely parameterized by

$$r_0 = a, \qquad r_3 = 1 - a, \qquad r_2 = \frac{a^2}{x} - 1, \qquad r_1 = \frac{(1-a)^2}{z} - 1, \qquad \sqrt{x} < a < 1 - \sqrt{z}. \tag{11}$$

The same probability region is obtained by allowing all directed nearest-neighbour chains with $0 < p_i < 1$.

Proof. Put $S = r_0 + r_3$. To leave vertex 1 last, the two nontrivial early explorers must occupy 3 and then 2. Their exit probabilities give

$$x = \frac{r_0^2}{S(S + r_2)}, \qquad z = \frac{r_3^2}{S(S + r_1)}. \tag{12}$$

With $a = r_0/S$, positivity of r_1, r_2 implies $x < a^2$ and $z < (1-a)^2$, proving the strict inequality in (10).

Conversely that inequality makes the interval in (11) nonempty. The specified resistances are positive, satisfy $S = 1$, and recover x, z by (12). Normalization recovers y. Scaling every realizing vector to $S = 1$ and solving (12) gives (11), which proves exhaustiveness and uniqueness of this parameterization.

For a directed chain set $a = 1 - p_0$. First-step conditioning on the two occupied-vertex arcs gives

$$x = \frac{a^2(1-p_3)}{1 - ap_3} < a^2, \qquad z = \frac{(1-a)^2 p_1}{1 - (1-a)(1-p_1)} < (1-a)^2.$$

The denominators are positive, and both inequalities are strict because $0 < a, p_1, p_3 < 1$. Thus directed chains satisfy the same necessary condition; the reversible construction already supplies every law satisfying it. $\square$

The condition in (10) is equivalently $y > 2\sqrt{xz}$. In particular the positive uniform law on the three possible last sites is not attainable with these dynamics.

5 A sharp five-cycle obstruction

Define the shifted last-site generating polynomial by

$$G(u) = \sum_{j=1}^{n-1} \mathbb{P}(L = j)\, u^{j-1}. \tag{13}$$

Theorem 7. *For $n = 3$ or $n = 4$, every positive nearest-neighbour chain on C_n has only negative real zeros in (13). On C_5, however, the positive edge resistances*

$$(r_0, r_1, r_2, r_3, r_4) = (10, 10, 1, 1, 20) \tag{14}$$

give

$$G(u) = \frac{1025 + 10575u + 13944u^2 + 4960u^3}{30504}, \tag{15}$$

which has exactly two nonreal zeros. Five is therefore the smallest cycle size allowing nonreal zeros, even if directed chains are admitted.

Proof. For $n = 3$, G is linear with positive coefficients. For $n = 4$, write $G(u) = x + yu + zu^2$. Theorem 6 gives $y^2 - 4xz > 0$, and the sum and product of its roots imply that both are negative.

For a transparent calculation of the witness, use the family $(r_0, r_1, r_2, r_3, r_4) = (t, t, 1, 1, 2t)$ with $t > 0$. The first right/left expansion probabilities are $2/3$ and $1/3$. After the next expansion, the three interval probabilities are

$$P_{2,0} = \frac{1}{3}, \qquad P_{1,1} = \frac{5t + 2}{3(3t + 1)}, \qquad P_{0,2} = \frac{t}{3(3t + 1)}.$$

One further application of (5) gives

$$\begin{aligned} D\mathbb{P}(L = 1) &= t^2(4t + 1), \\ D\mathbb{P}(L = 2) &= 2t(19t^2 + 21t + 5), \\ D\mathbb{P}(L = 3) &= (2t + 1)(8t + 3)(3t + 2), \\ D\mathbb{P}(L = 4) &= 2t(3t + 1)(3t + 2), \end{aligned} \tag{16}$$

where $D = 3(3t + 1)(3t + 2)(4t + 1)$. Their numerators sum to D. Setting $t = 10$ and cancelling a common factor of four yields (15). Equivalently the clockwise transition probabilities of the witness are $(2/3, 1/2, 10/11, 1/2, 1/21)$.

The discriminant of $au^3 + bu^2 + cu + d$ is

$$b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

For the primitive numerator of (15) it equals

$$-38\,803\,806\,734\,400 < 0.$$

A real cubic with negative discriminant has one real zero and a nonreal conjugate pair. All five conductances are positive, so this is not a limiting or directed-only example. The results for $n = 3, 4$ prove minimality in the number of vertices. $\square$

No minimality of the resistance ratio in (14) is claimed. The theorem refutes a natural extension of the familiar real-rooted Eulerian situation, not a conjecture formulated in the AIM notes.

6 Scope and reproducibility

The proof concerns finite-time, single-source, nearest-neighbour IDLA. It does not supply a law in higher dimensions or on a general longer-jump circulant. The two-boundary interval reduction is the essential restriction. The spline formulas and the homogeneous specializations are explicitly attributed; the note should not be read as a priority claim for those identities.

The accompanying standard-library Python code performs 23,967 exact rational assertions. It compares interval propagation, Cox–de Boor recursion, divided differences, and finite absorbing-chain linear systems. It includes 1,712 cycle cases, of which 963 use the independent absorbing-subset computation, 108 nonperiodic line environments, 81 four-cycle inverse fibers, and 18 permutation-enumerated q-Eulerian rows. The fixed cubic is checked by direct discriminant evaluation, a Sylvester determinant, and a Sturm count. Fresh isolated Python 3.9 and 3.13 runs agree apart from the interpreter version. These finite diagnostics are not the general proof or an independent review.

AI assistance was used in derivation, literature discovery, computation, and manuscript preparation. The argument has undergone an originating-researcher self-audit, not independent refereeing or formal proof-assistant verification. This is an unrefereed preprint.

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