

Essential Periods and Monochromatic Triangles in Measurable Layered Colorings

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Abstract

Let $f : \mathbb{R} \rightarrow \{-1, 1\}$ be measurable and color the plane by $\chi(x, y) = f(y)$. We say that an equilateral side length is *essentially avoided* if, for each fixed orientation, its monochromatic translation set has planar measure zero. If side s is essentially avoided, we prove that the essential period group of f is exactly $\sqrt{3}s\mathbb{Z}$. Thus at most one equilateral side length is essentially avoided. The classical EGMRSS side-length transfer then implies that every nonequilateral three-point configuration has a positive-measure set of monochromatic translations in some fixed orientation. The result includes collinear triples and survives arbitrary changes on a planar null set. The argument uses closed essential-period groups and one tilted equilateral triangle; it makes no interval or locally finite boundary assumption on the profile. It does not settle the conjecture for arbitrary planar two-colorings.

1 Statement and context

The planar two-color conjecture of Erdős, Graham, Montgomery, Rothschild, Spencer and Straus asks whether every nonequilateral set of three distinct points occurs monochromatically in every two-coloring of the plane. Congruence here includes reflection, and the three points may be collinear. By their side-length transfer, this is equivalent to the assertion that any fixed coloring avoids at most one equilateral side length; see [1, Lemma 1.3 and Corollary 1.4]. Alternating half-open horizontal stripes of height $\sqrt{3}/2$ show that a unit equilateral triangle can be avoided.

Jelínek, Kynčl, Stolař and Valla proved the conjecture for polygonal partitions and established a separate theorem for closed/open partitions [1]. Their polygonal definition requires locally finite boundary segments and a closure-of-interior condition. Currier, Moore and Yip proved the three-term arithmetic-progression case for arbitrary planar two-colorings [2]. Shkredov obtained results for measurable planar colorings under Bessel-function conditions on side ratios [3]. We instead assume that the coloring depends measurably on a single linear coordinate, without regularity of its one-dimensional color sets.

For a measurable coloring $\chi : \mathbb{R}^2 \rightarrow \{-1, 1\}$ and a fixed triple $V = (v_1, v_2, v_3)$, write

$$M_\chi(V) = \{u \in \mathbb{R}^2 : \chi(u + v_1) = \chi(u + v_2) = \chi(u + v_3)\}.$$

A side $s > 0$ is *essentially avoided* if $m_2(M_\chi(V)) = 0$ for every orientation of an equilateral triangle of side s . The orientation is fixed before the translation set is measured.

A coloring is *layered* if, after a Euclidean change of coordinates, it has the form $\chi_f(x, y) = f(y)$ for a Lebesgue-measurable $f : \mathbb{R} \rightarrow \{-1, 1\}$. It is *essentially layered* if it agrees with such a coloring outside a planar Lebesgue null set. We use completed Lebesgue measure, so arbitrary assignments on that null set are permitted. The essential period group of f is

$$P(f) = \{p \in \mathbb{R} : f(t + p) = f(t) \text{ for almost every } t\}.$$

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Theorem 1.1 (Period rigidity). *If χ_f essentially avoids side $s > 0$, then*

$$P(f) = \sqrt{3}s\mathbb{Z}.$$

Consequently a measurable layered coloring essentially avoids at most one equilateral side length.

Corollary 1.2. *Let χ be essentially layered. Every nonequilateral three-point configuration has a congruent copy V in a fixed orientation for which $m_2(M_\chi(V)) > 0$. In particular, every such configuration occurs monochromatically, including every three-point collinear configuration.*

The period conclusion is necessary for avoidance, not sufficient. We do not classify the profiles that attain their permitted exceptional side. The layering and measurability hypotheses are retained throughout.

2 Essential periods and a tilted triangle

Call h an *essential antiperiod* if $f(t+h) = -f(t)$ for almost every t . Each exceptional null set may depend on its shift.

Lemma 2.1. *For bounded measurable f , the group $P(f)$ is closed. If $P(f) = \mathbb{R}$, then f is constant almost everywhere. If f is not almost everywhere constant and $P(f)$ contains a nonzero number, then $P(f) = d\mathbb{Z}$ for a unique $d > 0$.*

Proof. Differences of periods are periods by translating the defining almost-everywhere equalities and intersecting finitely many conull sets. Suppose $p_n \in P(f)$ and $p_n \rightarrow p$. For each bounded interval I , continuity of translation in L^1_{loc} gives

$$\|f(\cdot + p) - f\|_{L^1(I)} \leq \|f(\cdot + p) - f(\cdot + p_n)\|_{L^1(I)} \rightarrow 0.$$

Hence $p \in P(f)$.

If every real number is a period, convolving f with a smooth compactly supported approximate identity gives continuous functions invariant under every translation, hence constants. Their local L^1 convergence forces those constants to converge on one bounded interval of positive length. The same limit equals f almost everywhere on every bounded interval.

Finally, a proper closed subgroup containing a nonzero element has a least positive element d . Otherwise arbitrarily small positive elements and their integer multiples approximate every real number, contradicting closedness and properness. The positive infimum is attained by closedness, and division with remainder gives $P(f) = d\mathbb{Z}$. $\square$

Proof of Theorem 1.1. Put $h = \sqrt{3}s/2$. For any fixed vertical offsets y_1, y_2, y_3 , the monochromatic translation set is

$$\mathbb{R} \times \{t : f(t+y_1) = f(t+y_2) = f(t+y_3)\}.$$

It is planar-null exactly when the second factor is one-dimensional-null, by the product measure formula on bounded rectangles.

The horizontal-base equilateral triangle has offsets $0, 0, h$. Its essential avoidance forces h to be an essential antiperiod. Thus $2h \in P(f)$, while $h \notin P(f)$; a nonzero-valued function cannot have the same shift as a period and an antiperiod. In particular, f is not almost everywhere constant. Lemma 2.1 gives

$$P(f) = d\mathbb{Z}, \quad 2h = Nd$$

with $d > 0$ and a positive integer N . The integer N is odd, since an even N would make h a period.

Suppose $N \geq 3$. Then $d = 2h/N < s$. With $X = \sqrt{s^2 - d^2} > 0$, consider the equilateral triangle

$$(0, 0), \quad (X, d), \quad \left(\frac{X}{2} - \frac{\sqrt{3}d}{2}, \frac{d}{2} + \frac{\sqrt{3}X}{2} \right). \quad (1)$$

Its third vertical offset is

$$c = \frac{d}{2} + \sqrt{h^2 - \frac{3d^2}{4}}.$$

Since $d \in P(f)$, its base colors agree for almost every vertical translate. Essential avoidance forces the apex color to be opposite almost everywhere, so c is another essential antiperiod. Comparing $f(t + c) = -f(t) = f(t + h)$ shows that $c - h \in P(f)$. Consequently

$$\frac{c - h}{d} = \frac{1 + \sqrt{N^2 - 3} - N}{2} \in \mathbb{Z}. \quad (2)$$

But for every integer $N \geq 3$,

$$(N - 1)^2 < N^2 - 3 < N^2, \quad 0 < \frac{1 + \sqrt{N^2 - 3} - N}{2} < \frac{1}{2},$$

contradicting (2). Thus $N = 1$ and $d = 2h$.

Two essentially avoided side lengths would give two least positive generators of the same group $P(f)$; they must be equal. $\square$

Only two fixed orientations are used in this proof. There is no intersection over an uncountable family of exceptional null sets. In particular, measurability is used precisely to close the essential-period group and exclude a nonconstant profile with all real shifts as essential periods.

3 The classical transfer in positive measure

For completeness we give coordinates for the finite EGMRSS side-length transfer, credited in [1, Lemma 1.3] to *Euclidean Ramsey theorems III*. This also makes its measure consequence transparent.

Let the target triple be $\{0, a, z\} \subset \mathbb{C}$, where $a > 0$ and $z \notin \{0, a\}$, and put $\omega = e^{i\pi/3}$. Set

$$\begin{aligned} B &= 0, & A &= a, & C &= (1 - \omega)a, & D &= z, \\ B' &= \omega z + (1 - \omega)a, & C' &= \omega z, & A' &= \omega(z - a), & D' &= \omega^2(z - a). \end{aligned}$$

The triangle ABC is equilateral of side a , and the six triples

$$\begin{aligned} (B, A, D), & \quad (C, A, B'), & (C, B, D'), \\ (B', A', D'), & (C', A', B), & (C', B', D) \end{aligned} \quad (3)$$

each have unordered squared side lengths $a^2, |z|^2, |z - a|^2$. These identities follow from $\omega^2 = \omega - 1$, $\bar{\omega} = 1 - \omega$ and $|\omega| = 1$; hence all six are congruent to the target.

If A, B, C are black and no triple in (3) is monochromatic, the first three force D, B', D' white. The fourth forces A' black and the fifth forces C' white, making the sixth all white, a contradiction. Interchanging colors handles the other base color. This remains valid when auxiliary points coincide, since such coincidences only restrict possible color assignments. Each target triple still has three distinct points. Collinear targets cause no exception.

For a fixed realization of the gadget, let $E = M_\chi(ABC)$ and let $M_1, \dots, M_6$ be the monochromatic translation sets of the six target triples. The color implication gives

$$E \subseteq \bigcup_{j=1}^6 M_j.$$

If $m_2(E) > 0$, choose a bounded square Q with $0 < m_2(E \cap Q) < \infty$. Some j then satisfies

$$m_2(M_j \cap Q) \geq \frac{1}{6}m_2(E \cap Q) > 0. \quad (4)$$

The base can first be aligned with any given equilateral triangle of side a by a Euclidean isometry.

Proof of Corollary 1.2. First take $\chi = \chi_f$. A nonequilateral triple has at least two unequal side lengths. By Theorem 1.1, at least one is not essentially avoided. There is therefore an equilateral triangle of that side in a fixed orientation whose monochromatic translation set has positive measure. The finite transfer and (4) give the required nonequilateral copy.

If $\chi = \chi_f$ outside a planar null set Z , then for every fixed triple V ,

$$M_\chi(V) \triangle M_{\chi_f}(V) \subseteq \bigcup_{j=1}^3 (Z - v_j).$$

The right side is null. Both essential avoidance and positive-measure translation sets are therefore unchanged. This proves the essentially layered case. $\square$

4 Scope and verification

The theorem covers arbitrary measurable profiles, including ones without locally finite boundary. It does not prove the unrestricted planar conjecture, remove measurability for layered colorings, or replace the side-ratio conditions in general measurable planar results. The period condition alone is not a classification of all avoiding profiles.

An accompanying standard-library checker verifies the squared distance identities symbolically over $\mathbb{Q}(\omega)[a, z, \bar{z}]$ and checks all 64 color assignments with monochromatic base. It includes seven exact target shapes, with collinear cases and coincident auxiliary points, 10,000 odd-integer checks of the strict square inequality, and period/antiperiod arithmetic on all 8,190 binary cyclic words of lengths at most 12. Incorrect-rotation and omitted-triple controls are detected. Normal and optimized runs agree. These are supplementary regressions; the universal measure and subgroup arguments are the written proofs above.

AI systems assisted with proof development, literature comparison, computation and writing. This is an unrefereed preprint; self-audit and exact tests are not independent peer review or proof-assistant verification. No absolute-priority claim is made.

References

- [1] V. Jelínek, J. Kynčl, R. Stolař and T. Valla, *Monochromatic triangles in two-colored plane*, *Combinatorica* **29** (2009), 699–718. doi:10.1007/s00493-009-2291-y. arXiv:math/0701940.
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- [3] I. D. Shkredov, *On some problems of Euclidean Ramsey theory* (2015). arXiv:1507.02727.