

Exact Progression Lengths in Finite Interval-Digit Sets

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Abstract

Let $D_{b,r,n}$ consist of the nonnegative integers with at most n base- b digits, all in $\{0, \dots, r\}$, where $0 \leq r \leq b-2$. We prove that its longest nonconstant arithmetic progression has exactly

$$(r+1) \max\{a \in \mathbb{N} : a \mid b^{n-1}, (b-r-1)a < b\}$$

terms, with the singleton case assigned length one. An explicit progression attains the bound. This finite-depth refinement of the orbit argument of Walker and Walker also gives the exact infinite-length limit, its first attaining depth, and a sharp least-prime-divisor criterion for the minimum possible value $r+1$. As an application, we determine the longest progression in every iterated sumset of a finite interval-digit cube. The contribution is a prescribed-alphabet and finite-depth refinement; the known one-missing-digit and prime-base constructions are explicitly credited.

1 Statement and relation to prior work

For a nonempty finite set $E \subset \mathbb{Z}$, let $\text{AP}(E)$ be the largest $L \geq 1$ for which $\{x+j\Delta : 0 \leq j < L\} \subset E$ with integers x and $\Delta > 0$. Thus a singleton has value one. For integers $b \geq 2$, $n \geq 1$, and $0 \leq r \leq b-2$, define

$$D_{b,r,n} = \left\{ \sum_{i=0}^{n-1} d_i b^i : d_i \in \{0, \dots, r\} \right\}.$$

Leading zeros are allowed. Put

$$q = b - r - 1, \quad B_n(b, r) = \max\{a \in \mathbb{N} : a \mid b^{n-1}, qa < b\}. \quad (1)$$

The maximum exists, since $a = 1$ is admissible and every admissible a is smaller than b/q .

Theorem 1. *For the above parameters,*

$$\text{AP}(D_{b,r,n}) = (r+1)B_n(b, r).$$

Writing $a = B_n(b, r)$, an extremizing progression is

$$\left\{ j \frac{b^{n-1}}{a} : 0 \leq j < (r+1)a \right\}.$$

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Walker and Walker [1] determined the largest progression length over all proper base- b Kempner sets. Their value is $(b-1)\beta(b)$, where $\beta(b)$ is the largest integer smaller than b whose prime factors divide b ; the alphabet $\{0, \dots, b-2\}$ attains it. Theorem 1 keeps both the alphabet width and the available number of digits as parameters. Its proof adapts their cyclic-orbit counting argument. Neither that method nor their limiting one-missing-digit formula is claimed as new here.

Prime-base restricted-digit constructions are also classical. In particular, Korsky [2, Lemma 6.1] records the progression-free property for a prime base and a short initial digit interval, with earlier Hilbert-cube context discussed there. Our treatment includes composite bases and quantifies the dependence on finite depth. A bounded literature search did not locate the formula (1) in this form; this is not a claim of certified priority.

2 Proof of the exact formula

We first isolate the elementary orbit facts used in the proof.

Lemma 2. *A translate of the grid of λ equally spaced points on $\mathbb{R}/\mathbb{Z}$ meets every half-open interval of length $\theta \in [0, 1]$ in at least $\lfloor \lambda\theta \rfloor$ points.*

Proof. Lift the interval to the real line and multiply coordinates by λ . Counting the grid points becomes counting integers in a half-open interval of length $\lambda\theta$, which gives either its floor or its ceiling. The same argument applies when the original interval crosses the chosen origin. $\square$

Lemma 3. *For a positive integer Δ , set*

$$\lambda_{-1} = 1, \quad \lambda_k = \frac{b^{k+1}}{\gcd(\Delta, b^{k+1})} \quad (k \geq 0).$$

Then $\lambda_k \mid b\lambda_{k-1}$ for every $k \geq 0$. Modulo b^{k+1} , the orbit of $x + j\Delta$ consists of a translate of a grid with λ_k points.

Proof. The additive order of Δ modulo b^{k+1} is λ_k . Since $\lambda_{k-1}\Delta$ is divisible by b^k , multiplication by b kills the orbit at the next level; hence its order divides $b\lambda_{k-1}$. This also covers $k = 0$. If $g = \gcd(\Delta, b^{k+1})$, the generated subgroup comprises the multiples of g . Translating it by x proves the grid assertion. $\square$

Proof of Theorem 1. For the lower bound, take any admissible divisor a in (1) and let $\Delta = b^{n-1}/a$. For $0 \leq j < (r+1)a$, the digit of $j\Delta$ in position $n-1$ is $\lfloor j/a \rfloor \leq r$, and all higher digits vanish. For a lower position $0 \leq k \leq n-2$, write $M = b^{k+1}$ and $g = \gcd(\Delta, M)$. Then

$$\begin{aligned} g &= \frac{\gcd(b^{n-1}, ab^{k+1})}{a} \\ &= \frac{b^{k+1}}{a} \gcd(b^{n-k-2}, a) \geq \frac{b^{k+1}}{a} > qb^k. \end{aligned}$$

The residue of $j\Delta$ modulo M is a multiple of g , so it is at most $M - g < (r+1)b^k$. Its digit in position k is therefore at most r . Thus every proposed term lies in $D_{b,r,n}$. When $n = 1$, there are no lower positions and $a = 1$. Taking $a = B_n(b, r)$ proves the lower bound.

For the upper bound, $r = 0$ is immediate. Assume $r \geq 1$, and let $x + j\Delta$, $0 \leq j < L$, be a progression in $D_{b,r,n}$, with $\Delta > 0$. If $L \leq r+1$, the desired bound follows from $B_n(b, r) \geq 1$. Otherwise $L \geq r+2$, and the diameter of the digit set gives

$$(r+1)\Delta \leq (L-1)\Delta \leq r \frac{b^n - 1}{b - 1}, \quad \Delta < \frac{b^n}{b - 1}.$$

Use the orders in Lemma 3. Since $\lambda_{n-1} \geq b^n/\Delta > b-1$, we have $\lambda_{n-1} \geq b$. There is consequently a first $k \in \{0, \dots, n-1\}$ with $q\lambda_k \geq b$. Minimality implies $q\lambda_{k-1} < b$; moreover, $\lambda_{k-1} \mid b^k \mid b^{n-1}$. Hence

$$\lambda_{k-1} \leq B_n(b, r), \quad \lambda_k \leq bB_n(b, r).$$

The forbidden digits in position k , after normalizing residues by b^{k+1} , form the half-open interval $[(r+1)/b, 1)$ of length q/b . Lemmas 2 and 3 show that a complete orbit contains at least $\lfloor q\lambda_k/b \rfloor \geq 1$ forbidden points. Our progression cannot traverse a complete orbit. Its L orbit points are therefore distinct and all allowed, giving

$$L \leq \lambda_k - \left\lfloor \frac{q\lambda_k}{b} \right\rfloor.$$

Finally, $f(t) = t - \lfloor qt/b \rfloor$ is nondecreasing on the nonnegative integers, because $0 < q < b$. It follows that

$$L \leq f(bB_n(b, r)) = (b-q)B_n(b, r) = (r+1)B_n(b, r).$$

The argument only bounds a run by the total number of allowed orbit points; it does not require those allowed points to be consecutive in orbit order. $\square$

3 Infinite depth and the sharp short-progression criterion

Let $\mathcal{K}_{b,r} = \bigcup_{n \geq 1} D_{b,r,n}$ and set

$$B_\infty(b, r) = \max\{a \in \mathbb{N} : qa < b, \text{ every prime factor of } a \text{ divides } b\}.$$

The integer one satisfies the prime-factor condition.

Corollary 4. *The maximum progression length in $\mathcal{K}_{b,r}$ is $(r+1)B_\infty(b, r)$. For a finite depth n , that value is attained if and only if $B_\infty(b, r) \mid b^{n-1}$. Writing v_p for the usual prime valuation, the first attaining depth is*

$$n_* = 1 + \max_{p \mid b} \left\lceil \frac{v_p(B_\infty(b, r))}{v_p(b)} \right\rceil.$$

Proof. Every divisor of a power of b has only prime factors dividing b , and every integer with that property divides some power of b . Every finite progression in $\mathcal{K}_{b,r}$ is contained in some $D_{b,r,n}$. Apply Theorem 1 and then the prime-factor criterion for divisibility by b^{n-1} . In particular, $B_n = B_\infty$ holds precisely when B_∞ itself is an admissible divisor at that depth. $\square$

The case $r = b-2$ recovers the Walker–Walker value $(b-1)\beta(b)$. The first-attaining-depth statement retains information lost in that limit.

Corollary 5. *Let p be the least prime divisor of b , and let $n \geq 2$. Then*

$$\text{AP}(D_{b,r,n}) = r+1 \iff r < b - b/p.$$

If the inequality fails, already $D_{b,r,2}$ contains $\{j(b/p) : 0 \leq j < p(r+1)\}$.

Proof. The smallest divisor greater than one of b^{n-1} is p . Thus $B_n = 1$ exactly when $qp \geq b$. Since b/p is an integer, this is equivalent to $r < b - b/p$. If $qp < b$, use $a = p$ in the lower-bound construction with $n = 2$. $\square$

For prime b , the inequality holds for every permitted r , so the value is $r + 1$ at every depth. Composite bases need not have this property. Some examples are:

b	r	q	n	$B_n(b, r)$	$\text{AP}(D_{b,r,n})$
6	4	1	2	3	15
6	4	1	3	4	20
7	4	2	2	1	5
8	4	3	2	2	10
10	8	1	2	5	45
10	8	1	3	5	45
10	8	1	4	8	72
10	7	2	3	4	32

The decimal row at depth four is the known 72-term progression from [1]; it is included to illustrate finite-depth stabilization, not as a new example.

4 Iterated sumsets and translated alphabets

For $0 \leq d \leq b - 1$, let

$$A_{b,d,n} = \left\{ \sum_{i=0}^{n-1} e_i b^i : 0 \leq e_i \leq d \right\},$$

and let sA denote the sum of $s \geq 1$ copies of a set A , allowing repetition.

Proposition 6. *Put $r = sd$. If $r \leq b - 2$, then*

$$\text{AP}(sA_{b,d,n}) = (r + 1) \max\{a \in \mathbb{N} : a \mid b^{n-1}, (b - r - 1)a < b\}.$$

If $r \geq b - 1$, then

$$sA_{b,d,n} = \left\{ 0, \dots, r \frac{b^n - 1}{b - 1} \right\}, \quad \text{AP}(sA_{b,d,n}) = r \frac{b^n - 1}{b - 1} + 1.$$

Proof. Every coefficient in $\{0, \dots, sd\}$ is a sum of s coefficients from $\{0, \dots, d\}$. These choices can be made independently in each position, and assembled into s elements of $A_{b,d,n}$. Consequently

$$sA_{b,d,n} = \left\{ \sum_{i=0}^{n-1} c_i b^i : 0 \leq c_i \leq r \right\}.$$

For $r < b$ there are no carries; the first assertion follows from Theorem 1. For $r \geq b - 1$, an induction on n shows that the displayed set is the full stated interval. Indeed, the interval for $n - 1$ positions has upper endpoint $r(b^{n-1} - 1)/(b - 1) \geq b^{n-1} - 1$; its translates by consecutive multiples of b^{n-1} overlap or are adjacent. $\square$

For binary digits, $d = 1$, this determines the longest progression in every s -fold sumset. For instance, base seven gives $\text{AP}(4A_{7,1,n}) = 5$ for all $n \geq 1$. The set $1 + A_{7,1,n} \subset [7^n]$ has 2^n elements, so its density in this interval is exactly $(2/7)^n$. Its fourfold sumset has a five-term progression and no six-term progression. The prime-base mechanism is known; the composite-base values and the exact finite formula supply the refinement.

This application is motivated by the example-building question in the frozen UnsolvedMath v1.6.0 corpus [3]. The density here tends to zero. We do not claim an optimal construction among all subsets of an interval, a result at arbitrary fixed positive density, a polynomial Bogolyubov counterexample, or resolution of that multi-part record.

Finally, allowing the same translated interval of digits $\{c, \dots, c + r\} \subset \{0, \dots, b - 1\}$ in all n positions translates $D_{b,r,n}$ by $c(b^n - 1)/(b - 1)$. Its longest progression length is therefore unchanged.

5 Reproducibility and limitations

The accompanying standard-library Python program constructs the finite digit sets directly and enumerates maximal positive-difference progression runs. It does not use the theorem to compute these maxima. Across 620 finite parameter choices, the maxima agree with Theorem 1. Separate exact-integer tests check 14,112 constructive progressions, comprising 1,013,834 terms; 38,439 translated-grid counts; and 12,096 instances of the sharp threshold. The archive also includes translated-alphabet and iterated-sumset checks. Replacing the strict inequality $qa < b$ by $qa \leq b$ fails on 75 tested instances. Normal and optimized Python executions produce identical reports.

These calculations are diagnostics; the proof is the argument above. They are not formal verification, independent review, or a certification of novelty. The manuscript was prepared with AI assistance for derivation, literature comparison, code, and exposition. It is an unrefereed, originating-author self-audit; its scope is the exact digit-set theorem.

References

- [1] A. Walker and A. Walker, Arithmetic Progressions with Restricted Digits, *American Mathematical Monthly* **127** (2020), no. 2, 140–150. doi:10.1080/00029890.2020.1682888. Preprint: [arXiv:1809.02430](https://arxiv.org/abs/1809.02430).
- [2] S. Korsky, Arithmetic Progression-Free Subset-Sum Sets, preprint (2026), version 1, [arXiv:2606.24139](https://arxiv.org/abs/2606.24139). Section 2.4 and Lemma 6.1.
- [3] ulamai, *UnsolvedMath*, frozen version 1.6.0, record AIM-COMBINATORICS-0025, commit c6e7f41b4eb6ce876feb83e7eb626be1b243fd7e. <https://huggingface.co/datasets/ulamai/UnsolvedMath>. The underlying AIM page is <http://aimpl.org/addcombapp/2/>; that live page could not be re-fetched during this audit.