

Boundary Rigidity for Equi-isoclinic Planes

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Abstract

Consider a family of real two-dimensional subspaces with common isoclinic projection length $1/2$. We prove that if one triangle of normalized cross-Gram matrices has negative determinant, then the entire family spans at most six dimensions and contains at most nine planes, independently of the initial ambient dimension. We give an exhaustive finite description relative to such a triangle and show that the nine-member families in this branch form one unordered congruence class. The proof uses the null relation of a singular anchor Gram matrix, eight possible extension types, and four explicit nonnegative polynomial identities. If no triangle has negative determinant, the familiar complex-model correspondence identifies the family with complex equiangular lines. The known maximum of nine planes in $\mathbb{R}^6$ is credited to Et-Taoui; the statement here concerns ambient collapse and the negative-triangle branch at the boundary parameter.

1 Statement and context

Let X_i be an orthonormal frame matrix for a real two-dimensional subspace U_i . An equi-isoclinic family with parameter $\alpha \in (0, 1)$ satisfies

$$X_i^T X_j = \alpha O_{ij}, \quad O_{ij} \in O(2) \quad (i \neq j).$$

Thus α is a projection length, rather than the squared parameter $\lambda = \alpha^2$ used in some references. Changing frames conjugates the triangle holonomy $O_{ij}O_{j\ell}O_{\ell i}$. Its determinant is therefore intrinsic. We call a triangle with determinant -1 a *reflection triangle*.

Theorem 1. *Let $U_1, \dots, U_n$ be equi-isoclinic real planes with parameter $\alpha = 1/2$. If the family contains a reflection triangle, then*

$$\dim(U_1 + \dots + U_n) \leq 6, \quad n \leq 9.$$

The bound nine is attained, and all unordered nine-member families satisfying this hypothesis are congruent. Every family with a fixed normalized reflection triangle has the finite description in Section 4.

The transition-matrix and holonomy framework, the classification of triples, and the correspondence between positive-triangle families and complex lines are established tools [2]. Et-Taoui proved that the maximum number of equi-isoclinic planes in $\mathbb{R}^6$ is nine [1]. We do not claim that maximum or the existence of a nine-plane configuration as new. Our argument proves an ambient-independent

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dimension collapse under the reflection-triangle hypothesis and gives a finite classification at the boundary value $\alpha = 1/2$.

Complex equiangular lines in dimension three have been studied in detail in [3]. Our reflection branch cannot be a complex realification. Recent dimension-counting estimates for general equi-isoclinic subspaces [4] remain quadratic in the ambient dimension for fixed subspace dimension. The theorem here does not prove the general fixed-angle linear bounds asked for in AIM Problem 4.1 of *High-dimensional phenomena in discrete analysis* [5]. In particular, it leaves the complex-line branch at $\alpha = 1/2$, smaller parameters, and higher-dimensional subspaces untreated.

2 A singular anchor and eight extension types

Choose a reflection triangle and label its frames X_1, X_2, X_3 . Normalize the two edges incident to X_1 to $I_2/2$. The remaining normalized edge is an orientation-reversing orthogonal matrix, hence is a reflection. A common conjugation diagonalizes it. We may therefore assume

$$X_1^T X_2 = X_1^T X_3 = \frac{1}{2}I_2, \quad X_2^T X_3 = \frac{1}{2}S, \quad S = \text{diag}(1, -1). \quad (1)$$

Write $X_i = (x_i, y_i)$. All x_i 's are orthogonal to all y_j 's, and their respective Gram matrices are

$$G_x = \frac{1}{2}(I_3 + J_3), \quad G_y = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & -1/2 \\ 1/2 & -1/2 & 1 \end{pmatrix}.$$

The eigenvalues of G_x are $2, 1/2, 1/2$; those of G_y are $3/2, 3/2, 0$. Thus

$$W = U_1 + U_2 + U_3, \quad \dim W = 5, \quad -y_1 + y_2 + y_3 = 0, \quad (2)$$

and

$$G_x^{-1} = 2I_3 - \frac{1}{2}J_3, \quad G_y^+ = \frac{4}{9}G_y,$$

where the latter denotes the Moore–Penrose inverse.

For any additional plane choose its frame X so that $X_1^T X = I_2/2$. Write

$$X_i^T X = \frac{1}{2} \begin{pmatrix} r_i \\ u_i \end{pmatrix}, \quad r_1 = (1, 0), \quad u_1 = (0, 1).$$

For each i , the rows r_i, u_i form an orthonormal basis of $\mathbb{R}^2$. Relation (2) gives $u_2 + u_3 = u_1$. Since the rows have unit length,

$$\begin{aligned} u_2 &= (\varepsilon\sqrt{3}/2, 1/2), & u_3 &= (-\varepsilon\sqrt{3}/2, 1/2), \\ r_2 &= (a/2, -a\varepsilon\sqrt{3}/2), & r_3 &= (b/2, b\varepsilon\sqrt{3}/2), \end{aligned} \quad (3)$$

where $\varepsilon, a, b \in \{1, -1\}$. For example, $\|u_1 - u_2\| = 1$ forces $\langle u_1, u_2 \rangle = 1/2$, which proves the asserted two possibilities for u_2, u_3 . There are exactly eight row types. Put $s = r_1 + r_2 + r_3$ and label them as follows.

Type	ε	a	b	s
0	+1	+1	+1	$(2, 0)$
1	+1	+1	-1	$(1, -\sqrt{3})$
2	+1	-1	+1	$(1, \sqrt{3})$
3	+1	-1	-1	$(0, 0)$
4	-1	+1	+1	$(2, 0)$
5	-1	+1	-1	$(1, \sqrt{3})$
6	-1	-1	+1	$(1, -\sqrt{3})$
7	-1	-1	-1	$(0, 0)$

Let R_{xi} and R_{yi} be the three-by-two row matrices of type i , and let $A = (x_1, x_2, x_3)$, $B = (y_1, y_2, y_3)$. The projection of its frame onto W is determined uniquely by the row data:

$$Y_i = \frac{1}{2}(AG_x^{-1}R_{xi} + BG_y^+R_{yi}). \quad (4)$$

The kernel compatibility $(-1, 1, 1)R_{yi} = 0$ ensures that the second term has the prescribed inner products with the y_j 's. Since $R_{xi}^T R_{xi} = R_{yi}^T R_{yi} = 3I_2/2$, and G_y^+ acts as $2/3$ on its range, we get

$$\begin{aligned} Y_i^T Y_i &= \frac{1}{4}(R_{xi}^T G_x^{-1} R_{xi} + R_{yi}^T G_y^+ R_{yi}) \\ &= I_2 - \frac{1}{8}s_i^T s_i. \end{aligned} \quad (5)$$

Call types 3 and 7 *passive*, since $s_i = 0$ and the whole plane lies in W . The other six types are *active*. For an active type $\|s_i\| = 2$; (5) shows that its perpendicular component has rank one. Thus its frame is necessarily

$$T_i = Y_i + \frac{1}{\sqrt{8}}z_i s_i, \quad z_i \in W^\perp, \quad \|z_i\| = 1. \quad (6)$$

Conversely this formula yields an orthonormal frame with the specified inner products against the anchors.

Each type can occur at most once. Indeed, if two planes had the same type, their projections Y_i would agree. For a unit v with $s_i v = 0$, (5) gives $\|Y_i v\| = 1$; both planes would contain $Y_i v$. This is impossible, since the projection length of a common unit vector would be one, rather than $1/2$. The same reasoning applies when the type is passive.

3 Pair identities and dimension collapse

For two distinct types define

$$D_{ij} = \frac{1}{4}(R_{xi}^T G_x^{-1} R_{xj} + R_{yi}^T G_y^+ R_{yj}), \quad E_{ij} = \frac{1}{8}s_i^T s_j.$$

For active types, (6) gives the cross Gram

$$M_{ij}(t) = D_{ij} + tE_{ij}, \quad t = \langle z_i, z_j \rangle.$$

If either type is passive then $E_{ij} = 0$, and t is irrelevant. Isoclinicity is equivalent to $M_{ij}(t)M_{ij}(t)^T = I_2/4$. Define its squared error

$$F_{ij}(t) = \|M_{ij}(t)M_{ij}(t)^T - I_2/4\|_F^2.$$

The row formulas (3) give all 28 pair identities in the following table; ij denotes the unordered pair of type indices.

Pairs	$F_{ij}(t)$
01, 02, 16, 25, 45, 46	$(t+1)^2(2t^2+3)/32$
All other active pairs	$(t^2-1)^2/16$
03, 17, 27, 35, 36, 37, 47	0
07, 13, 23, 34, 57, 67	5/8

These are polynomial identities, not sampled feasibility tests. To illustrate the substitution,

$$D_{01} = \begin{pmatrix} 1/2 & 0 \\ -\sqrt{3}/4 & 1/4 \end{pmatrix}, \quad E_{01} = \begin{pmatrix} 1/4 & -\sqrt{3}/4 \\ 0 & 0 \end{pmatrix},$$

which yields the first polynomial. Every other pair follows from the same two-by-two products with the displayed eight row types.

For any active pair, vanishing of F_{ij} forces $t \in \{-1, 1\}$. Unit vectors with this inner product are collinear. Consequently all nonzero residual directions lie on one common line, and

$$\dim(U_1 + \cdots + U_n) \leq \dim W + 1 = 6.$$

With no active type the dimension is five, and with only one active type the bound follows directly from (6).

4 Finite description and the sharp case

The preceding identities give both necessary and sufficient conditions. Choose a subset of the eight types and a sign $\sigma_i \in \{1, -1\}$ for every active type present. Use one unit $z \perp W$, and set $z_i = \sigma_i z$. The conditions are precisely:

1. On every present pair among 01, 02, 16, 25, 45, 46, one has $\sigma_i \sigma_j = -1$.
2. Type 3 is compatible only with active types 0, 5, 6.
3. Type 7 is compatible only with active types 1, 2, 4.
4. The two passive types are compatible with each other.

No other sign restriction is imposed. For any such choice, (4)–(6) and the pair identities construct the required family. Conversely, every family containing the normalized anchor triangle arises this way.

If neither passive type occurs, there are at most six additional planes. If exactly one occurs, there are at most three active types and thus four additional planes. If both occur, no active type is possible. Including the three anchors proves $n \leq 9$. There is also an independent short bound after the dimension collapse: the full Gram matrix G satisfies

$$\operatorname{tr} G = 2n, \quad \operatorname{tr} G^2 = 2n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+3).$$

The trace-rank inequality then gives

$$(2n)^2 \leq 6 \operatorname{tr} G^2 = 3n(n+3),$$

so $n \leq 9$.

Equality is attained by taking all six active types and no passive types, with

$$(\sigma_0, \sigma_1, \sigma_2, \sigma_4, \sigma_5, \sigma_6) = (1, -1, -1, -1, 1, 1).$$

The forced-negative graph is the connected even cycle

$$0 - 1 - 6 - 4 - 5 - 2 - 0.$$

Its valid signings are exactly the displayed signing and its simultaneous negative. Changing z to $-z$ removes this ambiguity. The anchor Gram fixes the anchors up to an ambient isometry, and then the projections and residuals are fixed. Hence any nine-member family with a reflection triangle is congruent, after ordering its remaining planes by type, to this construction. This proves the unordered uniqueness assertion.

The construction spans six dimensions, since its residuals are nonzero. Its Gram matrix is positive semidefinite by construction. Equality in the trace-rank inequality gives $G^2 = 3G$; thus the planes form a tight fusion frame in their six-dimensional span. They cannot be the realification of complex lines because their anchor triangle has negative determinant. The proof of Theorem 1 is complete.

5 The complex branch and the boundary restriction

For completeness, suppose all triangle determinants are positive. Normalize $O_{1i} = I_2$ for every i . Every remaining O_{ij} is then an orientation-preserving planar rotation. Thus the full Gram matrix commutes with

$$J = \text{diag}(J_2, \dots, J_2), \quad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Its kernel is J -invariant, and J descends to an orthogonal complex structure on the span of the frames. Each plane is a complex line; their complex inner products have modulus $1/2$. Conversely complex realifications have positive triangle determinants. This is the type-I correspondence of [2, Proposition 5.1].

Corollary 2. *At projection length $1/2$, every finite equi-isoclinic family of real planes either is a realification of complex equiangular lines, or contains a reflection triangle and therefore spans at most six dimensions and has at most nine members. These two branches are disjoint.*

For $\alpha > 1/2$, a reflection triangle is impossible: its Gram matrix restricted to the reflection eigenline has eigenvalue $1 - 2\alpha < 0$. The complex-model correspondence in this range is already part of [1]. For $\alpha < 1/2$, this eigenvalue is positive and the null relation (2) disappears. The finite-type argument does not apply there. Nor does it prove a linear bound in the complex branch at the boundary value.

6 Exact arithmetic supplement

The supplementary standard-library Python program works in $\mathbb{Q}(\sqrt{3})$ using rational coefficients. It verifies the eight projection identities and all 28 pair polynomial identities coefficient by coefficient. A separate enumeration checks 1,460 type/sign selections modulo common residual sign, of which 129 satisfy the pair conditions. These are anchored selections, not 129 distinct unordered congruence classes. Exactly one maximum signing remains after the common-sign convention.

The program also constructs the 18-by-18 witness Gram matrix, checks every off-diagonal block, computes rank six, and independently verifies $G^T = G$ and $G^2 = 3G$. The last two identities certify positive semidefiniteness without floating-point eigenvalue tolerances. The proof above does not rely on the enumeration: its type reduction and nonnegative identities are explicit.

A bounded literature review did not establish absolute priority for the precise reflection-branch statement. The historical nine-plane bound and the holonomy methods are credited above. This is an unrefereed scoped theorem, not a claimed resolution of the full AIM problem.

AI tools assisted with algebraic exploration, exact-check code, literature retrieval, and drafting. The author is responsible for the manuscript and its mathematical claims.

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