

A Counterexample to a Linear-Junta Inverse Theorem for Spherical Triangles

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Abstract

We construct smooth, real, mean-zero, 1-bounded functions on high-dimensional spheres whose average product over zero-sum unit triangles tends to $7/16384$, but whose correlation with every function of m linear coordinates is $O(\sqrt{m/d})$ when $m = o(d)$. The bound is uniform over all choices of coordinates and all test functions of L^2 norm at most one. The construction is a centered cosine of a balanced, rescaled quadratic form. A conditional coupling gives the projection estimate, and an elementary multivariate central limit argument computes the triangle average. This gives a negative answer to the dimension-free quantitative interpretation of Zakharov's spherical-triangle inverse question in the AIM problem list. It does not resolve the related extremal problem for zero-sum-free subsets of the sphere.

1 The question and the counterexample

Let σ_{d-1} be normalized surface measure on $S^{d-1} \subset \mathbb{R}^d$. For a real bounded measurable function f , define

$$T_d(f) = \mathbb{E}_{X+Y+Z=0} f(X)f(Y)f(Z), \quad X, Y, Z \in S^{d-1}.$$

The expectation uses the invariant probability measure on ordered zero-sum unit triangles. Equivalently, take a Haar orthonormal two-frame (U, V) and put

$$X = U, \quad Y = -\frac{1}{2}U + \frac{\sqrt{3}}{2}V, \quad Z = -\frac{1}{2}U - \frac{\sqrt{3}}{2}V. \quad (1)$$

The vectors have unit length and pairwise inner products $-1/2$.

Zakharov's Problem 2.4 in the AIM workshop list [1] asks whether $|f| \leq 1$ and $|T_d(f)| > \delta$ imply correlation with a function of $O_\delta(1)$ coordinates in some basis. We make the usual quantitative interpretation explicit: both the coordinate bound $m(\delta)$ and a positive correlation threshold $\varepsilon(\delta)$ should be independent of d . A *linear m -junta* here means a measurable function $g(x) = G(P_W x)$ for a subspace W of dimension at most m , where P_W is orthogonal projection. We will allow $\|g\|_2 \leq 1$, a larger class than $|g| \leq 1$.

For every even d , define

$$\begin{aligned} A_d &= \text{diag}(I_{d/2}, -I_{d/2}), & Q_d(x) &= \sqrt{d/2} x^\top A_d x, \\ t &= 2\sqrt{\log 2}, & \mu_d &= \mathbb{E} \cos(tQ_d(X)), & f_d(x) &= \frac{1}{2}(\mu_d - \cos(tQ_d(x))). \end{aligned} \quad (2)$$

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Theorem 1. *The functions f_d are smooth, real, even, mean-zero and 1-bounded. As $d \rightarrow \infty$ through even integers,*

$$T_d(f_d) \rightarrow \frac{7}{16384}, \quad \|f_d\|_2^2 \rightarrow \frac{225}{2048}. \quad (3)$$

For every m -dimensional subspace W with $0 \leq m < d$ and every $g(x) = G(P_W x)$ of $L^2(\sigma_{d-1})$ norm at most one,

$$|\mathbb{E}f_d(X)\overline{g(X)}| \leq 2\sqrt{2\log 2}\sqrt{m}\left(\frac{1}{\sqrt{d+2}} + \frac{1}{\sqrt{d-m}}\right). \quad (4)$$

In particular the supremum of these correlations tends to zero whenever $m = o(d)$, uniformly over the subspace and the test function.

Corollary 2. *For $\delta = 7/32768$, no dimension-free bounds $m(\delta) < \infty$ and $\varepsilon(\delta) > 0$ give the linear-coordinate inverse conclusion described above. The same failure holds for a dimension-free lower bound on normalized Pearson correlation.*

Indeed, the first limit in (3) is 2δ . For any proposed $m(\delta), \varepsilon(\delta)$ choose a sufficiently large even d so that $T_d(f_d) > \delta$ while the right side of (4) is less than $\varepsilon(\delta)$. It also bounds tests using fewer coordinates. For Pearson correlation, center the test function and normalize its L^2 norm. The denominator $\|f_d\|_2$ stays bounded away from zero by (3). Coordinates in a nonorthogonal linear basis are included: any rank- m linear map factors through projection onto its row space.

The statement is quantitative, not a claim of exactly zero correlation at each fixed dimension. The word “correlate” is not explicitly quantified in the source; a purely qualitative nonzero-correlation question would be different. The construction uses signed functions. The zero-sum-free set bounds of Dúcz [2] and Ge–Xu [3] concern a different extremal question, for which the hemisphere remains the motivating example.

2 A uniform estimate for all linear projections

The following elementary coupling is the main uniformity input.

Lemma 3. *Let $d \geq 2$, let A be real symmetric with $\|A\|_{\text{op}} \leq 1$, and put $Q(x) = \sqrt{d/2}x^\top Ax$. If $h : \mathbb{R} \rightarrow \mathbb{R}$ is L -Lipschitz and $F = h(Q) - \mathbb{E}h(Q)$, then for any m -dimensional subspace W , $0 \leq m < d$,*

$$\|\mathbb{E}[F(X) \mid P_W X]\|_2 \leq 2\sqrt{2}L\sqrt{m}\left(\frac{1}{\sqrt{d+2}} + \frac{1}{\sqrt{d-m}}\right). \quad (5)$$

No trace condition on A is required.

Proof. The case $m = 0$ follows from $\mathbb{E}F = 0$. Suppose $1 \leq m < d$ and write $P = P_W$, $P^\perp = I - P$. Let U, U' be independent copies of PX , and let V be an independent uniform unit vector in $W^\perp$. Put

$$s = \|U\|^2, \quad s' = \|U'\|^2, \quad r = \sqrt{1-s}, \quad r' = \sqrt{1-s'}, \quad X = U + rV, \quad X' = U' + r'V.$$

Both X and X' have law σ_{d-1} : conditional on its projection, a uniform sphere point has a uniform complementary direction. This also follows by normalizing independent Gaussian components in W and $W^\perp$. The coupled points need not be independent, but X' is independent of U . Thus $\mathbb{E}[F(X') \mid U] = 0$. Conditional Jensen gives

$$\|\mathbb{E}[F(X) \mid U]\|_2 \leq \|F(X) - F(X')\|_2 \leq L\|Q(X) - Q(X')\|_2. \quad (6)$$

The unscaled difference is

$$\begin{aligned} X^\top AX - X'^\top AX' &= U^\top AU - U'^\top AU' - (s - s')V^\top AV \\ &\quad + 2rU^\top AV - 2r'U'^\top AV. \end{aligned} \quad (7)$$

The usual sphere moments are

$$\mathbb{E}X_i X_j = \frac{\delta_{ij}}{d}, \quad \mathbb{E}X_i X_j X_k X_\ell = \frac{\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{il}\delta_{jk}}{d(d+2)}.$$

For a short derivation, write $a = \mathbb{E}X_1^4$ and $b = \mathbb{E}X_1^2 X_2^2$. Sign symmetry and a 45-degree rotation give $a = 3b$, while $1 = \mathbb{E}(\sum X_i^2)^2 = da + d(d-1)b$. For any symmetric B , contracting these moments yields

$$\text{Var}(X^\top BX) = \frac{2}{d(d+2)} \left(\text{tr}(B^2) - \frac{(\text{tr } B)^2}{d} \right). \quad (8)$$

Apply this with $B = PAP$. Its rank is at most m and its operator norm at most one. Independence of U, U' implies

$$\|U^\top AU - U'^\top AU'\|_2 \leq 2\sqrt{\frac{m}{d(d+2)}}.$$

Taking $B = P$ gives the same bound on $\|s - s'\|_2$, hence on the second term of (7), because $|V^\top AV| \leq 1$. Also $\mathbb{E}UU^\top = P/d$ and $\mathbb{E}VV^\top = P^\perp/(d-m)$. Independence gives

$$\mathbb{E}(U^\top AV)^2 = \frac{\text{tr}(PAP^\perp AP)}{d(d-m)} \leq \frac{m}{d(d-m)}. \quad (9)$$

The numerator is $\|P^\perp AP\|_{\text{HS}}^2 \leq m$. Since $r, r' \leq 1$, Minkowski's inequality in (7) now gives

$$\begin{aligned} \|Q(X) - Q(X')\|_2 &\leq \sqrt{d/2} \left(4\sqrt{\frac{m}{d(d+2)}} + 4\sqrt{\frac{m}{d(d-m)}} \right) \\ &= 2\sqrt{2m} \left(\frac{1}{\sqrt{d+2}} + \frac{1}{\sqrt{d-m}} \right). \end{aligned}$$

Together with (6), this proves the lemma. The case $d - m = 1$ uses a uniform direction on a two-point sphere and requires no change. $\square$

For (2), use $h(q) = -\cos(tq)/2$, whose Lipschitz constant is $t/2 = \sqrt{\log 2}$. Conditional expectation followed by Cauchy-Schwarz against $g = G(P_W X)$ proves (4). This argument is uniform in W ; no net of subspaces or regularity assumption on g is involved.

3 Quadratic statistics on a random triangle

Lemma 4. *For a Haar triangle (X, Y, Z) as in (1), the vector $(Q_d(X), Q_d(Y), Q_d(Z))$ converges in law, through even d , to a centered Gaussian vector (Z_1, Z_2, Z_3) with variances one and pairwise covariances $1/4$.*

Proof. Let G, H be independent standard Gaussian vectors in $\mathbb{R}^d$. Set

$$r = \|G\|, \quad k = \frac{G \cdot H}{r^2}, \quad \ell = \|H - kG\|, \quad U = G/r, \quad V = (H - kG)/\ell.$$

Almost surely these are defined, and (U, V) is a Haar orthonormal two-frame. Write $A = A_d$ and

$$S_d = \frac{G^\top AG}{\sqrt{2d}}, \quad R_d = \frac{H^\top AH}{\sqrt{2d}}, \quad B_d = \frac{G^\top AH}{\sqrt{2d}}.$$

These converge jointly to independent centered Gaussian variables S, R, B of variances $1, 1, 1/2$. Indeed, for $d = 2q$, put $J_i = (G_i^2 - 1, H_i^2 - 1, G_i H_i)$. The J_i are iid with mean zero and covariance $\text{diag}(2, 2, 1)$, and

$$(S_d, R_d, B_d) = \frac{1}{2\sqrt{q}} \sum_{i=1}^q (J_i - J_{q+i}).$$

The differences have covariance $\text{diag}(4, 4, 2)$ and finite moments. The ordinary multivariate central limit theorem proves the claim.

The law of large numbers gives $r^2/d \rightarrow 1$ and $\ell^2/d \rightarrow 1$ in probability, and $k = O_{\mathbb{P}}(d^{-1/2})$. For the second norm assertion, use $\ell^2 = \|H\|^2 - (G \cdot H)^2/r^2$ and $(G \cdot H)^2/r^2 = O_{\mathbb{P}}(1)$. The exact identities

$$\begin{aligned} \sqrt{d/2} U^\top AU &= \frac{d}{r^2} S_d, \\ \sqrt{d/2} V^\top AV &= \frac{d}{\ell^2} (R_d - 2kB_d + k^2 S_d), \\ \sqrt{d/2} U^\top AV &= \frac{d}{r\ell} (B_d - kS_d) \end{aligned}$$

and tightness of (S_d, R_d, B_d) allow Slutsky's theorem to be applied. Expanding the quadratic forms in (1) gives the limit

$$Z_1 = S, \quad Z_2 = \frac{1}{4}S + \frac{3}{4}R - \frac{\sqrt{3}}{2}B, \quad Z_3 = \frac{1}{4}S + \frac{3}{4}R + \frac{\sqrt{3}}{2}B. \quad (10)$$

Direct calculation gives variance one for each coordinate and covariance $1/4$ for every pair, as asserted. $\square$

4 The limiting triangle average

Put $C_i = \cos(tZ_i)$ and $c = \mathbb{E}C_i$. The Gaussian characteristic function and $t^2 = 4 \log 2$ give

$$c = e^{-t^2/2} = \frac{1}{4}, \quad \mathbb{E}C_i C_j = e^{-t^2} \cosh(t^2/4) = \frac{5}{64} \quad (i \neq j). \quad (11)$$

To compute the third moment, expand the product of cosines as the average of the exponentials with eight choices of signs. Two sign choices have variance $9/2$ for the signed sum of the Z_i , and the other six have variance $5/2$. Hence

$$\mathbb{E}C_1 C_2 C_3 = \frac{1}{4}e^{-9t^2/4} + \frac{3}{4}e^{-5t^2/4} = \frac{49}{2048}.$$

Centering and reversing the sign now give

$$\begin{aligned} \mathbb{E} \prod_{i=1}^3 (C_i - c) &= \frac{49}{2048} - 3 \left(\frac{1}{4} \right) \left(\frac{5}{64} \right) + 2 \left(\frac{1}{4} \right)^3 = -\frac{7}{2048}, \\ \mathbb{E} \prod_{i=1}^3 \frac{c - C_i}{2} &= \frac{7}{16384}. \end{aligned} \quad (12)$$

All test functions in these expectations are bounded continuous. Lemma 4 therefore implies their expectation convergence. In particular $\mu_d \rightarrow c$; replacing the centering constant by μ_d changes the bounded product expectation by a quantity tending to zero. This proves the first limit in (3). For the second, use

$$\mathbb{E}C_i^2 = \frac{1}{2}(1 + e^{-2t^2}) = \frac{257}{512}, \quad \text{Var}((c - C_i)/2) = \frac{1}{4} \left(\frac{257}{512} - \frac{1}{16} \right) = \frac{225}{2048}.$$

Finally (2) makes the smoothness, parity and exact centering of f_d immediate. Since μ_d and the cosine both lie in $[-1, 1]$, one has $|f_d| \leq 1$. Together with Lemma 3, this completes Theorem 1.

5 Scope and reproducibility

The obstruction is a bounded nonlinear function of a quadratic statistic involving many directions. Thus bounded harmonic degree and a bounded number of linear coordinates should not be conflated. The argument does not exclude inverse statements allowing bounded-degree polynomial statistics. It also makes no assertion about the planar or finite-field analogues mentioned in [1], or about the exact extremal measure of zero-sum-free sets.

The supplementary standard-library Python program checks the triangle Gram matrix, the limiting covariance, all eight Gaussian sign contributions, the rational moments in (11)–(12), and sample exact fourth-moment contractions. It produces identical results with ordinary and optimized Python execution. These checks support the arithmetic; the proofs above establish the limiting and all-subspace claims without numerical sampling.

This unrefereed note was developed with AI assistance in exploration, proof drafting and computational checks. The argument has undergone originating-model self-audit, not independent peer review or proof-assistant verification. The bounded literature search does not certify absolute priority.

References

- [1] D. Zakharov, Problem 2.4, *Inverse theorem for equilateral triangles on the sphere*, in *High-dimensional phenomena in discrete analysis*, AIM workshop problem list, May 2024. <http://aimpl.org/highdimdiscrete/2/>. Accessed 12 October 2026.
- [2] Á. Dúcz, *On the maximal measure of a spherical set avoiding solutions to $x + y + z = 0$* , arXiv:2607.05083v1, 2026. <https://arxiv.org/abs/2607.05083>.
- [3] H.-J. Ge and Z. Xu, *The sharp asymptotic density of zero-sum-free spherical sets*, arXiv:2607.05099v2, 2026. <https://arxiv.org/abs/2607.05099>.