

Sharp Interpolation for Equidegree Hilbert–Burch Curves

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Abstract

Let $\mathcal{H}_{t,a}$ be the Hilbert component of curves in $\mathbb{P}^3$ defined by the maximal minors of a $(t+1) \times t$ matrix of degree- a forms, over an algebraically closed field of characteristic zero. We prove that the largest number of general ambient points lying on a member of this component is

$$\iota(\mathcal{H}_{t,a}) = \frac{t(t+1)}{2} \left(\binom{a+3}{3} - 2 \right) = \frac{1}{2} \dim \mathcal{H}_{t,a}.$$

The lower bound is attained by smooth integral curves; the upper bound also covers the boundary of the component. A bidiagonal specialization splits the support into complete intersections indexed by intervals. The marked normal evaluation becomes two triangular systems with surjective diagonal blocks. For a general smooth member, evaluation of normal sections has maximal rank at any number of general points. The complete-intersection and linear cases are classical. This paper treats the equidegree family, not arbitrary ACM degree matrices or all components in AIM’s Hilbert-scheme interpolation question.

1 The interpolation problem and the result

Fix an algebraically closed field k of characteristic zero, and put $R = k[x_0, x_1, x_2, x_3]$. If $\mathcal{H}$ is an irreducible component of a Hilbert scheme of one-dimensional subschemes of $\mathbb{P}^3$, define $\iota(\mathcal{H})$ to be the largest integer r for which a general ordered tuple of r distinct points of $\mathbb{P}^3$ is contained in a member of $\mathcal{H}$. Here *general* means belonging to a nonempty Zariski-open subset of the ambient configuration space. The curve is allowed to vary with the tuple.

AIM’s 2010 workshop on components of Hilbert schemes asks for this invariant for arbitrary components [1, Problem 32]. An earlier related question for smooth ACM curves appears in Hartshorne–Sabadini–Schlesinger [3, §§6,8]. We determine the invariant for one specified infinite family.

For integers $t, a \geq 1$, let $\mathcal{H}_{t,a}$ be the closure of the locus of curves whose saturated ideal has resolution

$$0 \longrightarrow R(-(t+1)a)^t \xrightarrow{M} R(-ta)^{t+1} \longrightarrow I_C \longrightarrow 0. \quad (1)$$

Thus all entries of M have degree a and the maximal minors have height two. Ellingsrud’s results imply that this locus is smooth, irreducible and open in the Hilbert scheme [2, Theorems 1 and 2]. Consequently its closure is an irreducible component.

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Theorem 1. Set $A = \binom{a+3}{3}$ and $B = t(t+1)/2$. Then

$$\dim \mathcal{H}_{t,a} = 2B(A-2), \quad \iota(\mathcal{H}_{t,a}) = B(A-2). \quad (2)$$

Through a general tuple of $B(A-2)$ points there passes a smooth integral curve from the resolution locus (1). No subscheme on its Hilbert component, including a boundary member, contains a general tuple of $B(A-2) + 1$ points.

For a general smooth C in this component, write $\mathcal{N}_C = \mathcal{N}_{C/\mathbb{P}^3}$.

Corollary 2. For every $r \geq 0$ and a general reduced degree- r divisor Z on a general smooth $C \in \mathcal{H}_{t,a}$, the evaluation map

$$H^0(C, \mathcal{N}_C) \longrightarrow H^0(Z, \mathcal{N}_C|_Z)$$

has rank $\min\{2B(A-2), 2r\}$.

This is a statement about evaluation of global sections. It should not be confused with a definition of interpolation that additionally requires $H^1(\mathcal{N}_C) = 0$, such as [6, Definition 1.2]. We compute this cohomology in Section 5.

The case $t = 1$ is the classical equal-degree complete-intersection result [5, Propositions 5.1 and 5.2]. The case $a = 1$ is also classical: Perrin's Proposition 5.11 bis, crediting Ellia, gives $H^0(\mathcal{N}_C(-2)) = 0$, and his Proposition 5.6 gives the interpolation number. Kleppe–Miró-Roig [4, note to Theorem 5.3] also record Ellia's prior μ -semistability result for linear or quadratic determinantal curves. The uniform statement and triangular proof here are presented without an absolute-priority claim. In particular, the numerical examples below are not each claimed as new. Results for Brill–Noether general space curves [6] do not cover this entire family.

2 The component and a reducible specialization

Let V be the open subset of the affine space $\text{Mat}_{t+1,t}(R_a)$ on which the maximal minors have height two. It is nonempty by the construction below. Hilbert–Burch gives (1), and R/I_C is Cohen–Macaulay of dimension two. In particular, I_C is saturated. The corresponding projective family over V is flat: the fiberwise exact Hilbert–Burch complex consists of fixed free graded modules and gives the same Hilbert function on every fiber. It therefore induces a morphism to the Hilbert scheme.

Ellingsrud's dimension formula specializes to

$$\begin{aligned} D := \dim \mathcal{H}_{t,a} &= t(t+1)A - (t+1)^2 - t^2 + 1 \\ &= t(t+1)(A-2) = 2B(A-2). \end{aligned}$$

The negative terms account for changes of bases, with the common scalar acting trivially. The cited theorem, rather than parameter counting alone, identifies this number with the dimension of a smooth Hilbert locus. The distinct shifts ta and $(t+1)a$ leave no degree-zero cancellation. The Hilbert series

$$\frac{1 - (t+1)z^{ta} + tz^{(t+1)a}}{(1-z)^4}$$

also gives

$$d = \frac{t(t+1)a^2}{2}, \quad g = 1 + \frac{t(t+1)a^2((2t+1)a-6)}{6}. \quad (3)$$

There is a nonempty open subset $V^{\text{sm}} \subset V$ with smooth integral fibers. For completeness, evaluation of matrix coefficients at a fixed point of $\mathbb{P}^3$ is surjective. Locally trivializing $\mathcal{O}(a)$, the universal evaluation to the space of scalar matrices is a smooth morphism. The rank- $(t-1)$ stratum has codimension two and is smooth; the rank-at-most- $(t-2)$ stratum has codimension six. Its universal inverse image has dimension $\dim V - 3$, so a general matrix avoids it. For $t = 1$ this stratum is empty. After removing its closed image in V , the universal curve is smooth as a total space and remains proper over V . Generic smoothness in characteristic zero gives a smooth one-dimensional general fiber. Finally (1) implies $H^0(C, \mathcal{O}_C) = k$. A smooth such curve is connected and hence integral.

Choose general degree- a forms

$$A_1, \dots, A_t, B_1, \dots, B_t.$$

Use rows $0, \dots, t$ and columns $1, \dots, t$, and define M_0 by

$$(M_0)_{k-1,k} = A_k, \quad (M_0)_{k,k} = B_k, \quad (M_0)_{r,k} = 0 \quad (r \notin \{k-1, k\}). \quad (4)$$

Its maximal minors, up to signs, are

$$Q_k = A_1 \cdots A_k B_{k+1} \cdots B_t, \quad 0 \leq k \leq t.$$

Lemma 3. *The support of $C_0 = V(I_t(M_0))$ is*

$$\bigcup_{1 \leq i \leq j \leq t} C_{ij}, \quad C_{ij} = V(A_i, B_j).$$

Each C_{ij} is a smooth integral complete intersection of type (a, a) . Away from the zero sets of the other A 's and B 's, C_0 is locally equal, as a scheme, to C_{ij} .

Proof. Every Q_k vanishes on C_{ij} , since it contains A_i if $k \geq i$, and contains B_j if $k < j$. Conversely, at a common zero of all Q_k , let i be the first index with $A_i = 0$; such an index exists since $Q_t = 0$. Then $Q_{i-1} = 0$ forces $B_j = 0$ for some $j \geq i$.

Generality and Bertini give the assertions about the complete intersections. They are pairwise distinct and meet only in finitely many points. Thus the support is one-dimensional, the minors have height two, and $M_0 \in V$. At a point of C_{ij} where no other chosen form vanishes, Q_{i-1} is a unit times B_j , while Q_j is a unit times A_i . Every minor lies in (A_i, B_j) , which proves the local scheme equality. $\square$

Set $n = A - 2$. For every $i \leq j$, choose n distinct markings

$$p_{ij,1}, \dots, p_{ij,n} \in C_{ij}$$

away from all the other forms' zero sets, such that

$$R_a \longrightarrow \bigoplus_{\nu=1}^n \mathcal{O}_{C_{ij}}(a)|_{p_{ij,\nu}} \quad (5)$$

is surjective. To justify the choice, the Koszul resolution gives a surjection $R_a \rightarrow H^0(C_{ij}, \mathcal{O}_{C_{ij}}(a))$ with two-dimensional kernel, so the target has dimension $A - 2$. On an integral curve, a nonzero space of sections loses one dimension after imposing vanishing at a suitably general point. Inductively choose n such points. The full-rank condition is open, and excluding the finite forbidden sets does not change it. The resulting Bn markings on C_0 are all distinct and smooth.

3 The marked normal calculation

Fix a marking on C_{ij} and trivialize the fibers of $\mathcal{O}(a)$ there. The scalar matrix $M_0(p)$ has rank $t - 1$.

Lemma 4. *Its right kernel is spanned by a vector v with nonzero coordinates exactly in columns $i, \dots, j$. Its left kernel has a basis u^-, u^+ whose nonzero coordinates are exactly in rows $0, \dots, i - 1$ and $j, \dots, t$, respectively.*

Proof. Solve the two bidiagonal recurrences $M_0(p)v = 0$ and $uM_0(p) = 0$. All nonvanishing coefficients propagate a nonzero coordinate uniquely, up to scale. The only breaks are $A_i(p) = 0$ and $B_j(p) = 0$. The right recurrence consequently has one free interval, from i to j ; the left recurrence has the two stated free end intervals. This also proves the claim when $i = j$, $i = 1$, or $j = t$. $\square$

At a rank- $(t - 1)$ matrix, the normal space to the rank-deficient locus is $\text{Hom}(\ker M, \text{coker } M)$, of dimension two. In the bases of Lemma 4, a variation ΔM maps to

$$u^- \Delta M v, \quad u^+ \Delta M v. \quad (6)$$

This can be checked directly by putting M into a block form with an identity block of size $t - 1$ and a zero 2×1 block; the two entries of the remaining variation are the normal coordinates. By Lemma 3 and the smoothness of C_{ij} at the marking, the map from $\mathbb{P}^3$ to the scalar matrix space is transverse to the rank-deficient locus there. Hence (6) identifies the two normal directions of the curve, up to invertible choices of bases.

Partition all coefficient variations into two groups, each indexed by $1 \leq k \leq l \leq t$:

$$U_{kl} = \Delta M_{k-1,l}, \quad L_{kl} = \Delta M_{l,k}.$$

These groups are disjoint and exhaust the $t(t + 1) = 2B$ entries. Each entry can be chosen independently in R_a .

Lemma 5. *Coefficient variations surject onto the two normal directions at all Bn markings simultaneously.*

Proof. At a marking on C_{ij} , an upper entry contributes only to the minus coordinate in (6), and its coefficient is nonzero exactly when

$$k \leq i \leq l \leq j. \quad (7)$$

A lower entry contributes only to the plus coordinate, with nonzero coefficient exactly when

$$i \leq k \leq j \leq l. \quad (8)$$

These assertions follow by intersecting the row and column supports in Lemma 4. For example, an upper entry contributing to the plus coordinate would require $k - 1 \geq j$ and $l \leq j$, contrary to $k \leq l$. For a lower entry contributing to the minus coordinate one would need $l \leq i - 1$ and $k \geq i$, again impossible.

Order the pairs (i, j) by increasing $i + j$, with arbitrary tie-breaking. In (7), $k + l \leq i + j$, and equality forces $(k, l) = (i, j)$. The minus equations therefore form a block lower triangular system. The diagonal block for (i, j) is the map (5) for the single polynomial U_{ij} , multiplied at the markings by nonzero scalars. It is surjective. Choose the entries consecutively: earlier choices give known values, and U_{ij} corrects the remaining prescribed values on C_{ij} . Later entries do not affect earlier equations.

For the plus equations, use decreasing $i + j$. In (8), $k + l \geq i + j$, with equality only on the diagonal, so the same recursive argument applies to the L entries. The two systems are independent. Thus all $2Bn$ normal values can be prescribed. $\square$

4 Dominance and the sharp upper bound

Let W be the ordered Bn -marked incidence over V , restricted to distinct points smooth on their curve. It is an open subset of a fiber power of the smooth locus of the universal family. In particular, $W \rightarrow V$ is smooth of relative dimension Bn near our constructed marked curve, and W is smooth there.

Consider

$$\text{ev} : W \longrightarrow (\mathbb{P}^3)^{Bn}.$$

Lemma 5 supplies every normal direction in its differential. Moving each marking along its curve supplies the tangential directions. Thus $d\text{ev}$ is surjective at the constructed point. Since both source and target are smooth there, ev is smooth at that point and is dominant on the irreducible component W_0 through it.

The morphism $W_0 \rightarrow V$ is also dominant: near the constructed point W_0 is an open neighborhood in the smooth scheme W , and its projection to V is open. As V is irreducible, this image is dense. Consequently W_0 meets the inverse image of V^{sm} . This is a nonempty open subset of W_0 ; evaluation restricted to it remains dominant. Its constructible image contains a nonempty open subset of $(\mathbb{P}^3)^{Bn}$. This proves the asserted lower bound with smooth integral curves. The incidence differential is essential: merely placing markings on a special curve would not imply interpolation for general ambient points.

For the upper bound, let $\mathcal{U} \rightarrow \mathcal{H}_{t,a}$ be the universal family. Every fiber has support of dimension one, since its Hilbert polynomial is linear with positive leading coefficient. Thus the r -fold fiber product has dimension at most $D + r$. Its image in $(\mathbb{P}^3)^r$ cannot be dominant unless

$$3r \leq D + r, \quad \text{or equivalently} \quad r \leq D/2.$$

Embedded or isolated points on a boundary fiber do not increase this fiber dimension. With $D = 2Bn$, the upper bound proves Theorem 1 on the whole component.

Proof of Corollary 2. Surjectivity in Lemma 5 persists on an open neighborhood of the marked incidence, and hence occurs for markings on smooth fibers. Coefficient deformations map to embedded curve deformations. Therefore normal-section evaluation for such a smooth marked curve is surjective. On the smooth Hilbert locus,

$$h^0(C, \mathcal{N}_C) = \dim T_{[C]} \text{Hilb} = D = 2Bn.$$

Evaluation at Bn markings is consequently an isomorphism.

The full-rank locus is open in the marked family: on the smooth resolution locus, $h^0(\mathcal{N}_C) = D$ is constant and these spaces form a vector bundle compatible with base change. Its projection contains an open subset of the curve parameter space. For each curve there, the full-rank markings form a nonempty open, and thus dense, subset of its irreducible configuration space. Removing points from one isomorphism gives surjectivity for $r < Bn$; adding points gives injectivity for $r > Bn$. Openness gives the asserted statement for general markings in each degree. $\square$

5 Consequences, comparisons, and scope

For a smooth space curve, $\chi(\mathcal{N}_C) = 4d$. On the resolution locus, (3) therefore yields

$$h^1(C, \mathcal{N}_C) = t(t+1) \left(\binom{a+3}{3} - 2 - 2a^2 \right) = \frac{t(t+1)(a-1)(a-2)(a-3)}{6}. \quad (9)$$

Thus the normal bundle is nonspecial for $a = 1, 2, 3$, but special for every $a \geq 4$. Corollary 2 deliberately uses section evaluation rather than imposing nonspecialness. By Perrin’s Theorem 3.14 [5], it also says that the general normal bundle is h^0 -semistable, in his terminology.

Some values of (3) and (2) are:

t	a	d	g	$\dim \mathcal{H}_{t,a}$	$\iota(\mathcal{H}_{t,a})$
2	2	12	17	48	24
2	3	27	82	108	54
2	4	48	225	198	99
3	2	24	65	96	48
3	3	54	271	216	108

For instance, the first row is already a consequence of the quadratic μ -semistability result recorded in [4], combined with [5, Proposition 3.13 and Theorem 3.14], since $g = 17 \leq 2d = 24$. It is included as a consistency check, not as a new isolated claim. Many members of the family have $4d - 3g - 12 < 0$, so the general-moduli hypothesis of the Brill–Noether theorem in [6] does not apply.

Theorem 1 does not assert that every Hilbert component interpolates up to half its dimension. Plane conics already give a counterexample to that assertion: their component has dimension eight, but four general points of $\mathbb{P}^3$ are not coplanar. Nor does the theorem treat arbitrary nonuniform Hilbert–Burch degree matrices. The original AIM Problem 32 remains unresolved in that generality. In particular, the source is not narrowed to the locally Cohen–Macaulay locus as part of the statement.

6 Reproducibility

The accompanying source archive contains `check_equidegree.py` and deterministic receipts. The universal argument is the proof in Sections 2–4; no finite computation is used to infer its quantifiers. The script checks the kernel recurrences, support conditions, triangular orders, Hilbert-polynomial formulas and a plane-conic scope control. Normal and optimized Python runs each pass 309,988 structural assertions and produce identical receipts.

As an additional exact sanity check, the script constructs degree- a forms and distinct smooth markings over $\mathbb{F}_{101}$ for

$$(t, a) = (1, 1), (2, 1), (3, 1), (2, 2), (3, 2), (2, 3), (3, 3), (2, 4).$$

It assembles the entire coefficient-to-normal evaluation matrix and computes its rank by modular elimination. The ranks are respectively 4, 12, 24, 48, 96, 108, 216, 198, equal to the numbers of rows. All coefficients, markings, seeds and monomial exponents are retained for replay. These finite-field checks neither establish the characteristic-zero theorem by themselves nor amount to formal proof-assistant certification.

This manuscript was developed with AI assistance in mathematical exploration, drafting, literature comparison and computational checking. It is an unrefereed preprint; no independent human review or absolute priority certification is claimed.

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