

Frobenius Factorization Loci for Gauss Maps of Smooth Plane Curves

Alper Ferudun*

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Abstract

We construct canonical closed subschemes representing relative Frobenius factorization of a morphism from a smooth projective curve family, including over nonreduced bases. For Gauss maps of smooth plane curves of degree d , we identify these subschemes with explicit linear coefficient loci. For $q = p^e > 2$, the e th locus is empty unless $q \mid d - 1$; otherwise it is the smooth open in the space of equations $\sum_{i=0}^2 X_i R_i(X_0^q, X_1^q, X_2^q)$. The classification on geometric points is due to Pardini and Homma. Our scheme-level argument uses the pulled-back Euler sequence, Serre duality and a regular-sequence calculation to rule out infinitesimal thickenings. We obtain the dimensions and geometric irreducibility of the exact-height strata, treat the characteristic-two exception, and give explicit smooth families with height jumps. These results concern a specified part of the AIM Hilbert-scheme question; they do not construct characteristic-only components of space-curve Hilbert schemes.

1 Scope and notation

Problem 3 of the 2010 AIM Hilbert Schemes workshop list asks about the inseparable degrees of Gauss maps on loci of smooth embedded curves, their strata and Galois action, and the possibility of exotic Hilbert components in positive characteristic [1]. The corresponding frozen dataset record is AIM-ARITHMETIC_GEOMETRY-0050 [6]. We study smooth geometrically integral nonlinear curves, so that a function-field degree is defined, and compute the entire plane-curve case with a functorial scheme structure. The general higher-dimensional component question is not settled.

Throughout, p is prime. For a scheme C over a characteristic- p scheme S , write

$$C^{(q/S)} = C \times_{S, F_S^e} S, \quad F_{C/S}^e : C \longrightarrow C^{(q/S)}, \quad q = p^e.$$

This is *relative* Frobenius. In polynomial coordinates it raises the variables to q th powers while preserving coefficients in the base ring. The equation of the twisted source, however, has its coefficients raised to the q th power.

For a nonconstant morphism from a smooth integral curve over an algebraically closed field, its *height* is the integer $h \geq 0$ for which its inseparable degree onto its image is p^h . The normalization of the image can be used in this definition. No simultaneous normalization of a family will be needed.

The pointwise plane classification used below is classical: Homma's refinement of Pardini's theorem [3, Corollary 2.5] supplies the equations, and Kaji [4] supplies the relevant Gauss-map facts.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
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The additional issue addressed here is the scheme representing actual factorization after arbitrary base change. We prove the required infinitesimal assertion explicitly. A bounded primary-source comparison did not certify the novelty or absolute priority of this refinement.

2 Closed relative factorization loci

Theorem 2.1. *Let S be locally noetherian of characteristic p , let $\pi : C \rightarrow S$ be smooth and projective with geometrically integral one-dimensional fibers, let $Y \rightarrow S$ be smooth, and let $f : C \rightarrow Y$ be an S -morphism. For each $e \geq 0$ there is a canonical closed subscheme $T_e \subseteq S$ with the following property: an arbitrary base change $T \rightarrow S$ factors through T_e if and only if*

$$f_T = g_T \circ F_{C_T/T}^e \quad \text{for a morphism } g_T : C_T^{(p^e/T)} \longrightarrow Y_T \text{ over } T.$$

The factor g_T is unique. Moreover, $T_0 = S$, $T_{e+1} \subseteq T_e$, and the construction commutes with arbitrary base change. No reducedness assumption on S or T is required.

Proof. For a smooth curve family, degree-zero Cartier descent gives

$$\mathcal{O}_{C^{(p/S)}} = \ker(F_{C/S,*} \mathcal{O}_C \xrightarrow{d} F_{C/S,*} \Omega_{C/S}^1), \quad (1)$$

where the left-hand sheaf is embedded by relative Frobenius. Here is a local justification that includes nilpotent bases. For a characteristic- p ring A , the module $A[t]$ is free over $A[t^p]$ on $1, t, \dots, t^{p-1}$. Differentiation annihilates exactly the coefficient with exponent zero, since $1, \dots, p-1$ are units. Smooth curves have etale-local coordinates, and etale Frobenius squares are cartesian. Flat etale base change therefore gives (1). Relative Frobenius is finite locally free of rank p and a universal homeomorphism; see [5, Tags 0FW2 and 0EBS].

Consequently f factors through $F_{C/S}$ if and only if

$$df : f^* \Omega_{Y/S}^1 \longrightarrow \Omega_{C/S}^1$$

vanishes. For the nontrivial implication, an affine open of Y pulls back to an open of C corresponding to an open of $C^{(p/S)}$. Every pulled-back function descends uniquely by (1). The local ring homomorphisms glue. Injectivity in (1), or faithful flatness of Frobenius, proves uniqueness.

It remains to represent the condition that a vector-bundle section $s \in H^0(C, E)$ vanish identically on the relative curve. Work locally on S and choose a relatively ample line bundle L . For n sufficiently large, relative Serre vanishing and cohomology and base change give finite locally free sheaves

$$A = \pi_*(E^\vee \otimes L^n), \quad B = \pi_* L^n,$$

whose formation commutes with arbitrary base change, and a surjective evaluation $\pi^* A \rightarrow E^\vee \otimes L^n$. Contraction by s induces $a_s : A \rightarrow B$. After any base change, $s = 0$ implies $a_s = 0$. Conversely, if $a_s = 0$, the composite of the surjective evaluation with contraction is zero; hence contraction, and therefore s , is zero. The matrix entries of a_s define the required closed subfunctor. Uniqueness of the representing closed immersion glues these local constructions and makes them independent of n .

Apply this construction to $E = \mathcal{H}om(f^* \Omega_{Y/S}^1, \Omega_{C/S}^1)$ and $s = df$. It produces T_1 , on which f descends uniquely to a map on $C_{T_1}^{(p/T_1)}$. This source is again a smooth projective curve family. Repeat the construction for the differential of the descended map. Iteration gives T_e and its universal property. The construction and the unique descended maps commute with base change, as required. $\square$

Corollary 2.2. *Assume in addition that every geometric fiber map is nonconstant. Then the geometric points of T_e are precisely the fibers of height at least e . If A_Y is relatively ample and $\deg f_s^* A_Y = D > 0$ is constant, then T_e is empty unless $p^e \mid D$. In particular $T_e = \emptyset$ for $e > \nu_p(D)$. The exact-height schemes $Z_e = T_e \setminus T_{e+1}$ are locally closed, and height is upper semicontinuous.*

Proof. A function field of one variable over an algebraically closed field has degree p over its subfield of p th powers. A nonconstant map has zero differential precisely when its function-field image lies in that subfield. Iteration proves the height assertion, including when a separable part is present; compare [5, Tag 0CCV]. A relative p^e -Frobenius factorization on a geometric fiber gives $D = p^e \deg g_s^* A_Y$. The remaining claims follow from Theorem 2.1. In particular height may increase, not decrease, under specialization. $\square$

For a smooth family of embedded curves of degree d and genus g , put $L = \mathcal{O}_C(1)$. The embedding supplies a surjection onto the rank-two first-principal-parts bundle:

$$H^0(\mathbb{P}^r, \mathcal{O}(1)) \otimes \mathcal{O}_C \twoheadrightarrow P_{C/S}^1(L).$$

This quotient classifies the tangent-line Gauss map γ to the Grassmannian. The principal-parts sequence therefore gives

$$\gamma^* \mathcal{O}_{\text{Gr}}(1) = \det P_{C/S}^1(L) = L^{\otimes 2} \otimes \omega_{C/S}, \quad D = 2d + 2g - 2. \quad (2)$$

For nonlinear embedded curves the Gauss map is nonconstant. Thus the smooth geometrically integral nonlinear Hilbert locus has the finite canonical stratification of Corollary 2.2.

If the family and map are defined over $\mathbb{F}_p$, then T_e and Z_e are Galois stable. On an irreducible component height has a generic minimum attained on a dense open. Geometric components of a finite-type $\mathbb{F}_p$ -scheme have finite arithmetic Frobenius orbits, and generic height is constant along an orbit. This need not fix each individual component.

3 The plane loci as schemes

Let k be algebraically closed of characteristic p , and let $H_d \subset \mathbb{P}(k[X_0, X_1, X_2]_d)$ be the open scheme of smooth plane curves of degree $d \geq 2$. Every such curve is geometrically integral. Take T_e from Theorem 2.1 for its universal Gauss map.

Theorem 3.1. *For $q = p^e > 2$, one has an equality of closed subschemes of H_d :*

$$T_e = \begin{cases} H_d \cap \mathbb{P}(L_{d,q}), & q \mid d-1, \\ \emptyset, & q \nmid d-1, \end{cases}$$

where, for $d = qm + 1$,

$$L_{d,q} = \left\{ \sum_{i=0}^2 X_i R_i(X_0^q, X_1^q, X_2^q) : R_i \in k[U_0, U_1, U_2]_m \right\}.$$

The nonempty scheme is smooth and geometrically irreducible of dimension $3\binom{m+2}{2} - 1$. In characteristic two the exceptional first locus is $T_1 = H_d$, without a restriction on d .

Proof. We first specify the classical input. Homma [3, Section 2 and Corollary 2.5] identifies the indicated geometric point set: for $q > 2$, generic tangent contact at least q is equivalent to the displayed coefficient condition. When the contact order exceeds two, it is the inseparable Gauss degree. In characteristic two every smooth plane curve has inseparable Gauss map; see [4, Corollary 2.3 and its following remark]. The separable degree for smooth plane curves is one by [4, Corollary 4.5]. These pointwise classifications are credited inputs.

Suppose $d = qm + 1$, and set $W = H_d \cap \mathbb{P}(L_{d,q})$. The monomials in $L_{d,q}$ are distinct: their exponent vectors have one entry congruent to one modulo q and the other two congruent to zero. Hence their number is $3\binom{m+2}{2}$. The open W is nonempty because it contains the Fermat equation, and it is smooth of dimension

$$w = 3\binom{m+2}{2} - 1. \quad (3)$$

The universal gradient on W is $[R_0(X^q) : R_1(X^q) : R_2(X^q)]$. It factors through relative q -Frobenius even over nonreduced test schemes. Thus $W \subseteq T_e$ is a closed scheme inclusion.

Fix a geometric point $[F] \in W$, with $F = \sum_i X_i R_i(X^q)$. The $R_i(X^q)$ have no common projective zero: Euler's identity reads $F = \sum_i X_i F_{X_i}$, so such a zero would be singular on C_F . Surjectivity of Frobenius shows that the three degree- m forms $R_i(U)$ also have no common projective zero. They consequently form a homogeneous regular sequence in $k[U_0, U_1, U_2]$.

Write $D = C_F^{(q/k)}$. The descended map is

$$g : D \longrightarrow \mathbb{P}^2, \quad [U] \longmapsto [R_0(U) : R_1(U) : R_2(U)], \quad g^* \mathcal{O}(1) = \mathcal{O}_D(m).$$

The equation of D raises the coefficients of F to the q th power; the coefficients of the displayed R_i are unchanged. Pulling back the Euler sequence gives

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_D(m)^{\oplus 3} \longrightarrow g^* T_{\mathbb{P}^2} \longrightarrow 0. \quad (4)$$

We claim that the associated map $H^1(\mathcal{O}_D) \rightarrow H^1(\mathcal{O}_D(m))^{\oplus 3}$ is injective. By Serre duality its dual is multiplication by the R_i :

$$H^0(\mathcal{O}_D(d-3-m))^{\oplus 3} \longrightarrow H^0(\mathcal{O}_D(d-3)).$$

These spaces are restrictions of homogeneous polynomials, since both degrees are nonnegative and less than d . The quotient by the regular sequence (R_0, R_1, R_2) has Hilbert series $(1+t+\cdots+t^{m-1})^3$, and therefore vanishes in degrees above $3m-3$. Since

$$d-3 = qm-2 > 3m-3 \quad (q \geq 3), \quad (5)$$

the multiplication map is surjective. This proves the claim. It follows from (4) that every first-order deformation of g with fixed source is represented by changing its three degree- m polynomial coordinates; scalar multiples of the original triple are the kernel.

An arbitrary tangent vector to T_e at $[F]$ is a deformation $F + \epsilon G$ over $k[\epsilon]/(\epsilon^2)$, modulo kF . Its relative q -Frobenius-twisted source is the constant family $D_{k[\epsilon]}$, since $\epsilon^q = 0$. The universal property of T_e gives a deformation of g on this fixed source. By the preceding argument it has coordinates $R_i(U) + \epsilon Q_i(U)$ for degree- m forms Q_i .

After pullback, this triple and the gradient of $F + \epsilon G$ define the same projective morphism using the same line bundle $\mathcal{O}_{C_{F+\epsilon G}}(d-1)$. They differ by a unit in $k[\epsilon]$: a smooth proper family with geometrically integral fibers has only base constants as global functions. Absorbing the scalar into the Q_i gives

$$G_{X_i} = Q_i(X^q)$$

as homogeneous polynomials. Indeed restriction in degree $d - 1 < d$ to the relative degree- d hypersurface is injective, also over the dual numbers. Euler's identity, with $d = 1$ in k , now yields

$$G = \sum_i X_i G_{X_i} = \sum_i X_i Q_i(X^q) \in L_{d,q}.$$

Hence $T_{[F]}T_e = T_{[F]}W$, of dimension w .

Homma's theorem gives the common support W , so every closed local ring of T_e has dimension and embedding dimension both equal to w . It is regular. Over the perfect field k this makes T_e smooth and reduced, and its support then identifies it with W . If $q \nmid d - 1$, the absence of geometric points implies emptiness. For $p = 2$, the closed subscheme T_1 has the whole underlying space of the reduced scheme H_d as its support; its ideal is contained in the nilradical of H_d and is therefore zero. $\square$

The theorem descends to $\mathbb{F}_p$ by faithful flatness. Thus it also describes the locus after any extension of that field.

4 Exact heights and component action

Corollary 4.1. *Put $N = \binom{d+2}{2} - 1$ and $Z_e = T_e \setminus T_{e+1}$. If p is odd, Z_0 is nonempty, smooth and geometrically irreducible of dimension N . For $e \geq 1$, Z_e is nonempty exactly when $p^e \mid d - 1$, and its dimension is*

$$3 \binom{(d-1)/p^e + 2}{2} - 1.$$

If $p = 2$, Z_0 is empty, Z_1 is nonempty, smooth and geometrically irreducible of dimension N , and the same divisibility and dimension rule holds for $e \geq 2$. Every nonempty stratum has one geometric component, fixed by arithmetic Frobenius.

Proof. The successive nonempty coefficient loci in Theorem 3.1 are properly nested linear spaces, since their dimensions strictly decrease. Each has a nonempty smooth open. Removing the next proper linear subspace leaves a nonempty geometrically irreducible open of the asserted dimension. The first locus in odd characteristic is also a proper linear subspace when nonempty. In characteristic two apply the same argument to $H_d \setminus T_2$. All equations are defined over $\mathbb{F}_p$, proving the component statement. $\square$

For $d = q + 1$ and $q > 2$, the locus in Theorem 3.1 is the family $F_B = (X^{[q]})^\top B X$ with B invertible. Its parameter scheme is the determinant complement in $\mathbb{P}^8$, isomorphic as a variety to PGL_3 . This statement concerns coefficient parameters, not a free coordinate-change orbit: a Fermat curve has a nontrivial stabilizer. Projective equivalence of smooth Frobenius forms is classical; see, for example, [2, Theorem 2.1.1]. For $p = 2$ and $d = 3$, by contrast, $T_1 = H_3$ has dimension nine. The eight-dimensional matrix family omits the possible $X_0 X_1 X_2$ term and is not the whole first locus.

5 Explicit smooth families with height jumps

Proposition 5.1. *Let p be odd and $q = p^e \geq 3$. The plane family*

$$F_\lambda = X^{q+1} + Y^{q+1} + Z^{q+1} + \lambda X^2 Y^{q-1}$$

is smooth exactly when $1 + 2\lambda^{(q+1)/2} \neq 0$. On this smooth parameter set, the height is e at $\lambda = 0$ and zero elsewhere.

Proof. The gradient is

$$(X^q + 2\lambda XY^{q-1}, \quad Y^q - \lambda X^2 Y^{q-2}, \quad Z^q).$$

A common projective zero has $Z = 0$ and $XY \neq 0$. Putting $u = X/Y$ gives $\lambda u^2 = 1$ and $u^{q-1} + 2\lambda = 0$, equivalent over the algebraic closure to $1 + 2\lambda^{(q+1)/2} = 0$. Conversely a square root of $1/\lambda$ gives such a zero when that condition holds. Euler's identity places the zero on the curve, proving the smoothness criterion.

At zero the height is exactly e by Theorem 3.1; q divides $d - 1 = q$ but pq does not. For $\lambda \neq 0$, the cross monomial violates the residue pattern modulo p , so that theorem gives height zero. There is also a direct differential check. On $Z = 1$ the Hessian applied to $v = (F_Y, -F_X)$ is

$$2\lambda Y^{q-3}(Y, -X)(Y F_Y + X F_X) = -2\lambda Y^{q-3}(Y, -X)$$

on the curve, generically nonzero for $\lambda \neq 0$. The third gradient coordinate is one on this chart, so this also verifies separability of the projective Gauss map. $\square$

Proposition 5.2. *In characteristic two, for $q = 2^e \geq 4$ the family*

$$F_\lambda = X^{q+1} + Y^{q+1} + Z^{q+1} + \lambda X^3 Y^{q-2}$$

is smooth for all λ . Its height is e at zero and one at every nonzero parameter.

Proof. The gradient $(X^q + \lambda X^2 Y^{q-2}, Y^q, Z^q)$ has no common projective zero. It factors one relative Frobenius, with residual coordinates

$$(U^{q/2} + \lambda UV^{q/2-1}, \quad V^{q/2}, \quad W^{q/2}).$$

The twisted source has coefficient λ^2 in its cross term; the displayed map uses λ . On $W = 1$, the differential of the first residual coordinate is $\lambda V^{q/2-2}(V dU + U dV)$. Its value on the tangent vector given by the two partial derivatives of the twisted equation is $\lambda V^{q/2-2}$, by Euler's identity. It is generically nonzero when $\lambda \neq 0$. Thus the residual map is separable and the original height is one. At zero the height is exactly e , as in Proposition 5.1. Equivalently, the nonzero cross term violates the modulo-four coefficient condition of Theorem 3.1. $\square$

The scheme issue cannot in general be reduced to inspecting geometric fibers. For $A = k[\epsilon]/(\epsilon^2)$ and $e \geq 1$, consider

$$\mathbb{P}_A^1 \longrightarrow \mathbb{P}_A^1, \quad [X : Z] \longmapsto [X^{p^e} + \epsilon X Z^{p^e-1} : Z^{p^e}].$$

Its geometric fiber has height e , but its relative differential on the affine chart is $\epsilon dt \neq 0$. It does not factor even one relative Frobenius over A . This is an example about morphisms, not a counterexample to Theorem 3.1; it explains the need for the tangent-space calculation in that theorem.

6 Relation to the Hilbert component question

The smooth plane-curve parameter scheme is a nonempty open of projective coefficient space over the integers, and each characteristic fiber is irreducible. It is horizontal. A smooth plane curve can also be lifted by lifting coefficients to Witt vectors; its discriminant remains a unit. Accordingly the height jumps above do not exhibit a characteristic-only Hilbert component.

The dimensions and component geometry for arbitrary space curves in $\mathbb{P}^r$, $r \geq 3$, are not determined here, nor do we construct a vertical smooth space-curve component. Results on possible Gauss fields when the embedding varies, such as [7, Corollary 3.2 and Theorem 3.3], address a different issue and do not by themselves answer this fixed-component question.

The work was developed with AI assistance and a detailed self-audit. It has not been independently reviewed or proof-assistant formalized. The supplementary exact polynomial, monomial, rank and exceptional-case checks are finite regression tests, not substitutes for the general proofs. The pointwise classification is credited to earlier literature, and no absolute-priority claim is made for the scheme-level refinement.

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