A Generically Nonreduced Component for Hilbert Function (1,4,10,10)

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Abstract

We give a computer-assisted proof that the very-compressed locus with Hilbert function $(1,4,10,10)$ is the reduced support of a generically nonreduced irreducible component of $\mathrm{Hilb}^{25}(\mathbb{A}^4)$ over an algebraically closed field of characteristic zero. The support is $\mathbb{A}^4 \times \mathrm{Gr}(10,20)$ and has dimension 104. This supplies the case $a = 10$ omitted from Jelisiejew's published theorem for $(1,4,10,a)$, $a = 6,7,8,9$, and identifies the generic component throughout the interval in AIM Problem 31. The new computation gives 244 primary-obstruction quadrics in 46 normal variables over $\mathbb{F}_3$. Exact F4 rounds produce a positive pure leading power of every variable, proving that the normal obstruction algebra is zero-dimensional. A properness argument transfers this certificate to characteristic zero, where the published Białynicki–Birula criterion applies. The general obstruction framework and the four earlier cases are established prior work. We do not compute the generic nilpotent local algebra or claim absolute priority.

1 The question and the result

Problem 31 on page 4 of the 2010 AIM workshop list *Components of Hilbert schemes* asks for the generic component containing a general Artinian algebra with Hilbert function $(1,4,10,a)$, for $6 \le a \le 10$ [1]. Nonsmoothability alone does not identify that component. Jelisiejew proved componenthood and generic nonreducedness for $a = 6,7,8,9$ [3, Theorem 1]. He explicitly left $a = 10$ open, pointing to its 50 negative tangent directions [3, Section 1.2].

Throughout the theorem, k is algebraically closed of characteristic zero, $S = k[x_1,x_2,x_3,x_4]$, and S_d is its degree-d part. Write $\mathfrak{m} = (x_1,x_2,x_3,x_4)$. The fixed-support very-compressed locus consists of

$$I_W = W + \mathfrak{m}^4, \qquad W \in \mathrm{Gr}(20-a, S_3). \tag{1}$$

Here the Grassmannian parametrizes subspaces of dimension $20-a$. It is isomorphic to $\mathrm{Gr}(a,20)$. Moving the support gives a smooth, closed irreducible locus

$$L_a \simeq \mathbb{A}^4 \times \mathrm{Gr}(20-a, S_3) \simeq \mathbb{A}^4 \times \mathrm{Gr}(a,20). \tag{2}$$

The closedness and the very-compressed description are part of the general framework of [3, Sections 1.2 and 3].

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Theorem 1.1. *The locus L_{10} is the reduced support of an irreducible component of* $\mathrm{Hilb}^{25}(\mathbb{A}^4)$*. Its dimension is* 104*, and the component, with the scheme structure inherited from the Hilbert scheme, is generically nonreduced.*

Corollary 1.2. *For every $6 \le a \le 10$, L_a is the reduced support of a generically nonreduced irreducible component of* $\mathrm{Hilb}^{15+a}(\mathbb{A}^4)$ *of dimension $4 + a(20 - a)$. Put $K_a = k(L_a)$ and let $t = (t_1, \ldots, t_4)$ be the generic support. The topological generic point represents the algebra*

$$K_a[y_1, y_2, y_3, y_4]/\big(W_\eta + (y_1, y_2, y_3, y_4)^4\big), \qquad y_i = x_i - t_i, \tag{3}$$

where W_η is the generic $(20 - a)$-plane of cubic forms. In particular, $\mathrm{trdeg}_k\, K_a = 4 + a(20 - a)$*.*

The corollary combines Theorem 1.1 with the published four cases; those cases are not claimed as new. The Hilbert-scheme local ring at its generic point is an Artinian nonreduced local ring with residue field K_a. Formula (3) describes the length-$(15 + a)$ algebra parametrized by that point, not the multiplication table or length of the Hilbert scheme's nilpotent generic local ring.

2 Primary obstructions and translations

We use the scalar $\mathbb{G}_m$-action on affine four-space and the corresponding grading of the Hilbert tangent space. At a homogeneous point $[Z] = [\mathrm{Spec}(S/I_W)]$, put

$$T = \mathrm{Hom}_S(I_W, S/I_W), \qquad T_{<0} = \bigoplus_{d<0} T_d.$$

On the cubic-generated open set, multiplication $S_1 W \to S_4$ is onto, hence $I_W = (W)$. The witness below lies in this open set.

Let π be the map from the positive Białynicki–Birula attractor to its fixed locus. When $T_{\le -2} = 0$, the primary obstruction criterion of [3, Proposition 6] gives a surjection onto the coordinate ring of $\pi^{-1}([Z])$ from the quadratic algebra

$$P_Z = \frac{\mathrm{Sym}^\bullet(T_{<0}^\vee)}{\mathrm{im}(\mu^\vee)}. \tag{4}$$

Here μ is the degree--2 primary obstruction arising from Yoneda multiplication; evaluating it on a universal first-order deformation gives its homogeneous quadratic equations. Our new input is an exact computation that cuts the normal part of this algebra down to dimension zero.

Four tangent directions come from translations. For a length-25 scheme, the barycentre

$$b(Z) = \frac{1}{25}\big(\mathrm{Tr}_{\mathcal{O}_Z}(x_1), \ldots, \mathrm{Tr}_{\mathcal{O}_Z}(x_4)\big) \tag{5}$$

satisfies $b(Z + u) = b(Z) + u$. Translation and the centred slice therefore give a product decomposition. This works both in characteristic zero and in characteristic three. In compatible tangent coordinates, the primary-obstruction algebra has a four-variable translation factor. The computation verifies that its quadratic equations contain neither a pure translation term nor a translation-normal cross term.

For very-compressed homogeneous ideals the positive-degree Hilbert tangents vanish: a positive-degree homomorphism sends the cubic generators to the zero part $(S/I_W)_{\ge 4}$. The local Białynicki–Birula comparison used in the proof of [3, Theorem 1] thus identifies an open Hilbert-scheme neighbourhood with the corresponding attractor. Its fixed locus is a Grassmannian chart. Once its fibres are set-theoretically just translation orbits on a nonempty open chart, (2) is the support of a component. We verify the required open support statement in Section 6.

3 Direct construction of the normal equations

Choose cubic generators $f_1, \ldots, f_{10}$ of I_W. A degree--1 tangent is represented by quadratic tails $q_i \in S_2$ satisfying the syzygies of these generators. The obstruction equations are obtained by elementary syzygy lifting, without calling a deformation package.

For a linear or quadratic syzygy $r = (r_i)$, solve

$$\sum_i r_i q_i + \sum_i h_i f_i = 0. \tag{6}$$

The h_i are constants for a linear syzygy and linear forms for a quadratic syzygy. The second-order error is

$$\sum_i h_i q_i. \tag{7}$$

Its class is taken modulo all corrections $\sum_i r_i \ell_i$, where ℓ_i are arbitrary linear second-order tails. In degree five it is also projected to $(S/I_W)_3$. Applying these operations to a universal linear combination of tangent vectors, and symmetrising, produces the quadratic primary obstruction in (4).

Only syzygies of total degrees four and five can contribute: the obstruction has degree -2, so a relation of total degree at least six maps to $(S/I_W)_{\geq 4} = 0$. Both the space of all degree-five syzygies and the quotient by linear multiples of degree-four syzygies are computed. This avoids discarding necessary corrections by using an incomplete presentation.

The implementation `primary_obstruction_linearization.py` performs these operations by exact modular row reduction. An independently implemented tangent and syzygy calculation is in `generic_linear_algebra.py`. The former was calibrated against the four published cases: on deterministic samples, the normal quadratic ideal is supported only at the origin for each $a = 6, 7, 8, 9$. This calibration checks the implementation; the proof for $a = 10$ uses its own explicit witness below.

4 A characteristic-three witness

The ordered cubic monomial basis is

$$\begin{aligned}(x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1x_2^2, x_1x_2x_3, x_1x_2x_4,\\ x_1x_3^2, x_1x_3x_4, x_1x_4^2, x_2^3, x_2^2x_3, x_2^2x_4,\\ x_2x_3^2, x_2x_3x_4, x_2x_4^2, x_3^3, x_3^2x_4, x_3x_4^2, x_4^3).\end{aligned} \tag{8}$$

Over $\mathbb{F}_3$, take W to be the row span of

$$\begin{pmatrix}
0 & 2 & 1 & 2 & 2 & 0 & 2 & 2 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 1 & 2 & 0 & 2 & 0 \\
0 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\
2 & 0 & 0 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
2 & 1 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
1 & 2 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 2 & 1 & 0 & 2 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 2 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 0 & 1 & 2 & 0 & 1 & 0 \\
0 & 1 & 0 & 2 & 2 & 2 & 1 & 0 & 2 & 2 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & 0 \\
2 & 1 & 2 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 2 & 2 & 0 & 2 \\
1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 & 1 & 0 & 0 & 2 & 1 & 1 & 2 & 1
\end{pmatrix}. \tag{9}$$

The exact row-reduction data are recorded in Table 1. In particular, the resulting quotient has Hilbert function $(1, 4, 10, 10)$ and length 25.

Table 1: Exact data for the matrix in (9).

Quantity	Value
Rank of W	10
Rank of $S_1 W \to S_4$	35
Linear syzygies	5
All degree-five syzygies	44
Minimal degree-five syzygies	24
$\dim T_{-1}$	50
$\dim T_{\leq -2}$	0
Translation rank	4
Second-order target dimension	490
Correction-map rank	40
Obstruction cokernel dimension	450
Independent normal quadrics	244
Normal variables	46

After splitting off translations, let

$$J \subset \mathbb{F}_3[z_1, \ldots, z_{46}]$$

be the ideal of the 244 independent normal quadrics. The complete equations, in the variable order used below, are provided as `a10_primary_obstruction_p3.ms` in the reproducibility supplement. They contain 136,542 nonzero terms. Equation generation, its structural ranks, and the matrix in (9) are retained in machine-readable form; checksums are given in Appendix A.

5 The finite zero-dimensionality calculation

Use homogeneous graded reverse lexicographic order with $z_1 > \cdots > z_{46}$. Exact F4 row reductions through degree four produce elements of J whose leading monomials include

$$z_i^2 \ (1 \leq i \leq 21), \qquad z_i^3 \ (22 \leq i \leq 35), \qquad z_i^4 \ (36 \leq i \leq 45). \tag{10}$$

The archived partial leading basis has 34,964 entries: 244 in degree two, 2,837 in degree three and 31,883 in degree four. At this stage no pure power of z_{46} is available. In particular, stopping at degree four does not establish zero-dimensionality. The intermediate leading-ideal Hilbert values through degree five are $(1, 46, 837, 6588, 1810, 1810)$.

Starting in degree five, exact pairs were processed in batches of at most 1,000. The first batch added 1,000 rows; the second added 810 rows and had 190 zero reductions. After that completed exact F4 round, the solver produced the one-term polynomial

$$z_{46}^5 \in J. \tag{11}$$

The preserved detector output records characteristic three, 46 variables, degree five and polynomial length one. A repeat run from the original equations used a different degree-five batch size, 2,000. It reproduced the lower-degree data, then processed a 4973×4783 degree-five matrix, adding 1,810 rows with 190 zero reductions, and produced the same one-term witness. The two detector records agree byte for byte.

Proposition 5.1. *The exact computation* (10)–(11) *implies that* $\mathbb{F}_3[z_1, \ldots, z_{46}]/J$ *is zero-dimensional.*

Proof. Let M be the monomial ideal generated by the 46 powers in (10) and (11). Each generator is the leading monomial of a polynomial in J, so $M \subseteq \operatorname{in}(J)$. A positive power of every variable belongs to M. Thus its quotient is spanned by finitely many monomials, and so is the quotient by $\operatorname{in}(J)$. The ideal J has the same Hilbert function as its initial ideal. Its quotient is consequently finite-dimensional over $\mathbb{F}_3$. $\square$

This implication does not require a complete Gröbner basis or the length of the quotient. The partial leading monomials and last-variable witness are sufficient when they are obtained from exact ideal-preserving row reductions.

6 Characteristic zero and the component argument

Interpret (9) as an integer matrix and work over $\mathbb{Z}_{(3)}$. The pivots used for the tangent chart, syzygy lifts, correction quotient, translation splitting and selected obstruction equations are nonzero modulo three. The corresponding formulas therefore lift over this local base. The selected 244 normal equations specialize to J; they are equations of the full primary obstruction, whether or not additional equations occur in characteristic zero.

Lemma 6.1. *The characteristic-zero normal primary-obstruction algebra at the lifted point is zero-dimensional. The same conclusion holds on a nonempty open set of* $\mathrm{Gr}(10, S_3)$ *in characteristic zero.*

Proof. The common projective zero scheme of the 244 homogeneous equations is a closed subscheme of $\mathbb{P}^{45}_{\mathbb{Z}_{(3)}}$. By Proposition 5.1, its closed fibre is empty. Its image in $\operatorname{Spec} \mathbb{Z}_{(3)}$ is closed by properness. If that image contained the generic point, it would contain the closed point as well. Hence the generic fibre is empty. This gives zero-dimensionality over $\mathbb{Q}$, and over any characteristic-zero extension field. Additional obstruction equations can only shrink the zero scheme.

Next choose an affine Grassmannian neighbourhood U on which the required Gaussian pivots remain invertible. The selected normal equations vary regularly there. Their projective common zero scheme in $\mathbb{P}^{45} \times U$ is proper over U and has an empty fibre at the lifted point. Removing its closed image from U gives a nonempty open neighbourhood on which every normal projective fibre is empty, as required. □

Proof of Theorem 1.1. On the open set furnished by Lemma 6.1, the quadratic algebra (4) is supported on the four translation coordinates. Its quotient, the Białynicki–Birula fibre, is therefore supported on the translation axis. Conversely, that fibre contains the four-dimensional translation orbit. Its reduced support is precisely that orbit. This uses the support of the normal homogeneous algebra, not merely an equality of dimensions.

The local comparison in Section 2 identifies the union of these fibres with an open Hilbert-scheme neighbourhood whose underlying space is $\mathbb{A}^4 \times U$. Since L_{10} is closed and irreducible, it is the reduced support of an irreducible component, of dimension $4 + 10 \cdot 10 = 104$.

The tangent estimate in [3, Lemma 1] gives at least 50 degree--1 tangent directions on the cubic-generated open set for this Hilbert function. Only four are tangent to the reduced translation fibre; the degree-zero tangent space of the Grassmannian has dimension 100. Thus the Hilbert tangent space has dimension at least 150, strictly more than the dimension 104 of the smooth reduced support. This strict excess persists throughout a nonempty open set on the component. Over the perfect field k, a generically reduced component would have a nonempty smooth open set away from other components. The strict excess excludes that possibility, proving generic nonreducedness. □

Proof of Corollary 1.2. Apply Theorem 1.1 for $a = 10$ and [3, Theorem 1] for $a = 6, 7, 8, 9$. The dimension and the represented generic algebra follow from the product description (2) and the universal subspace in (1). □

7 Reproducibility and limitations

The companion archive includes the equation generator, independent structural linear algebra, explicit input, partial leading-basis output, last-variable witnesses, terminal transcripts, parser and narrowly scoped local patch to msolve 0.10.1 [2]. The patch exposes a degree guard, controls pair-batch size beginning at a specified degree, and reports a one-term last-variable power only after a completed exact F4 row reduction. It does not replace the field arithmetic or reduction algorithm.

The guard and detector were checked on $(x^2, xy+y^2) \subset \mathbb{F}_3[x, y]$: degree two does not provide a pure y power, while degree three produces y^3. The characteristic-three equation system was regenerated byte for byte. Structural data were also checked on samples over $\mathbb{F}_{101}$ and $\mathbb{F}_{65537}$. These finite-field comparisons are consistency checks; the general characteristic-zero implication is the properness argument above, not an extrapolation from finitely many primes.

Both decisive F4 runs use the same patched msolve implementation. The independent parser checks the recorded monomials, witness metadata and hashes; it does not supply a separate ideal-membership proof kernel. A supplementary Singular 4.4.1 run was stopped after more than 90 minutes without output and is not used as evidence for the theorem. Accordingly this is a computer-assisted proof with an explicit software trust boundary, not a Lean formalization or a cross-engine verification of the full computation. No full Gröbner basis, obstruction-algebra length, or generic nilpotent multiplication table is claimed.

The source and literature search is bounded to the primary material recorded in the accompanying review, with cutoff September 5, 2026. It located the published four-case theorem and its explicit remaining case; it does not prove that no later solution exists. The framework and all four earlier cases are credited to Jelisiejew. This manuscript is an unrefereed preprint and makes no absolute priority claim.

AI systems assisted with research, computation, proof development and manuscript preparation. The author is responsible for the submitted text and for any subsequent corrections. This disclosure does not constitute external mathematical peer review.

A Identification of the computation files

The following SHA-256 values identify the exact artifacts used in the computer-assisted argument. Paths are relative to `reproducibility/checks/`. The full list, software provenance and unsuccessful attempts are preserved in the companion archive.

Equation system. `a10_primary_obstruction_p3.ms`
`76e0a8150451ddde52b6db1dda02b5523b35cf24a218f67a4170e3abd53a5c64`

Degree-four leading basis. `a10_primary_obstruction_p3.degree4.leading.out`
`85b821e0b8efc6f18ab47d7e15f37c3c72c587a54184ae441c0a6cd139da66c7`

Degree-five witness. `a10_primary_obstruction_p3.degree5.pure_last_power.txt`
`52bd88828bf1d4c8fff4b98321b6f016b719f98e32694593b9e7bc82271d490e`

Repeat transcript.
`a10_primary_obstruction_p3.degree5.reproduction.transcript.txt`
`956912738ac8ee6105c6beef66e648c5d6f7751ab7a9dfe792d93d0df32a9b0a`

References

[1] American Institute of Mathematics. Components of Hilbert schemes. Workshop problem list, 2010. Problem 31, p. 4.

[2] Jérémy Berthomieu, Christian Eder, and Mohab Safey El Din. msolve: A library for solving polynomial systems. In *Proceedings of the 2021 International Symposium on Symbolic and Algebraic Computation*, pages 51–58. ACM, 2021.

[3] Joachim Jelisiejew. Generically nonreduced components of Hilbert schemes on fourfolds. *Rendiconti del Seminario Matematico, Università e Politecnico di Torino*, 82(1):171–184, 2024.