A Seventeen-Dimensional Component of the Hilbert Scheme of Eighteen Points on an Integral Curve

Alper Ferudun*

September 5, 2026

Abstract

We give an explicit integral projective curve whose Hilbert scheme of eighteen points has a rational component of dimension seventeen. This answers affirmatively Problem 20 in the 2010 AIM workshop list *Components of Hilbert Schemes*, even with the curve required to be integral and projective. The curve comes from the known numerical semigroup $\langle 13, 14, 15, 16, 17, 18, 21, 23\rangle$ of Herzog–Kumashiro–Stamate. A five-generator ideal of colength eighteen is a canonical module; its full Hilbert tangent space is the seventeen-dimensional normalization quotient. A flat family of truncated unit translates identifies an open subset of the Hilbert scheme with $\mathbb{A}^{17}$. Two independent finite syzygy certificates verify the tangent calculation over every field. In characteristic zero, a separate application of Kass's theorem gives a component of dimension $d-1$ for every $d \geq 18$ on the same curve. The semigroup, canonical-module identities and general moduli theorem are established prior work; the point is their explicit Hilbert-scheme application. No absolute priority or minimal-length claim is made.

1 The question and the example

Problem 20 on page 3 of the 2010 AIM workshop problem list *Components of Hilbert Schemes* asks whether the Hilbert scheme of length-d subschemes of a curve can have an irreducible component of dimension strictly less than d [1]. The source's opening conventions specify irreducible components. An adjacent remark records a nonsmoothable component of dimension equal to its length. The question does not explicitly impose reducedness, projectivity, or a ground-field characteristic. Our example is integral and projective, so it does not depend on admitting embedded points into the meaning of a curve.

Let k be an algebraically closed field, initially of arbitrary characteristic. Set

$$S = \langle 13, 14, 15, 16, 17, 18, 21, 23\rangle, \qquad R = k[t^S], \qquad I = (t^{26}, t^{27}, t^{29}, t^{31}, t^{32}) \subset R. \tag{1}$$

Here $k[t^S]$ denotes the subalgebra of $k[t]$ spanned by the monomials whose exponents belong to S. Consider the degree-27 monomial map $\mathbb{P}^1 \to \mathbb{P}^{10}$ with coordinates

$$X_a = u^{27-a}v^a, \qquad a \in \{0, 13, 14, 15, 16, 17, 18, 21, 23, 26, 27\}. \tag{2}$$

Let C be its scheme-theoretic image.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: `https://github.com/AlperTheKing`.

Theorem 1.1. *The curve C is integral and projective, with normalization $\mathbb{P}^1$ and exactly one singular point. Its affine chart $X_0 \neq 0$ is* $\operatorname{Spec} R$. *The ideal I defines a length-*18 *subscheme $Z \subset C$ supported at that singular point, and*

$$\dim_k T_{[Z]} \operatorname{Hilb}^{18}(C) = 17.$$

There is an open immersion $\mathbb{A}^{17} \hookrightarrow \operatorname{Hilb}^{18}(C)$ whose image contains $[Z]$. Its closure is a rational irreducible component of dimension 17. *These assertions hold in every algebraically closed characteristic.*

Corollary 1.2. *If* $\operatorname{char} k = 0$, *then for every $d \geq 18$ the same curve C has an irreducible component of* $\operatorname{Hilb}^d(C)$ *of dimension $d - 1$.*

The semigroup in (1) is not new. It is the type-5 example in Herzog–Kumashiro–Stamate [3, Example 5.4(iii)], already present in the 2021 preprint, and is reused in Herzog–Kumashiro [2, Example 2.5]. The latter work records numerical canonical-trace data that motivate the calculation below. Corollary 1.2 is an application of Kass's general Hilbert-component theorem [4, Theorem 2.7(3)], not a new general moduli theorem. The direct tangent computation and the explicit unit family provide a local description independent of that application.

2 The semigroup and projective curve

The values of S below its conductor are

$$S \cap [0, 25] = \{0, 13, 14, 15, 16, 17, 18, 21, 23\}. \tag{3}$$

Every integer from 26 through 38 is a sum of the generators:

$$\begin{gathered} 26 = 13 + 13, \quad 27 = 13 + 14, \quad 28 = 13 + 15, \\ 29 = 13 + 16, \quad 30 = 13 + 17, \quad 31 = 13 + 18, \\ 32 = 14 + 18, \quad 33 = 15 + 18, \quad 34 = 13 + 21, \\ 35 = 14 + 21, \quad 36 = 13 + 23, \quad 37 = 14 + 23, \quad 38 = 15 + 23. \end{gathered}$$

Adding 13 inductively proves that every integer at least 26 belongs to S. Since 25 is absent, the conductor is 26 and the Frobenius number is 25. The set of gaps is

$$\mathcal{G} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 19, 20, 22, 24, 25\}. \tag{4}$$

Lemma 2.1. *The curve C has the geometric properties in Theorem 1.1, and its arithmetic genus is* 17.

Proof. The endpoint coordinates in (2) never vanish simultaneously on $\mathbb{P}^1$, so the map is everywhere defined. Its scheme-theoretic image is integral and projective. The chart $X_0 \neq 0$ has coordinate ring R: adding the generators t^{26} and t^{27} changes nothing, since $26 = 13 + 13$ and $27 = 13 + 14$. On the chart $X_{27} \neq 0$, all coordinate ratios are nonnegative powers of u/v, and $X_{26}/X_{27} = u/v$. This chart is therefore $\operatorname{Spec} k[u/v]$. These two charts cover C, since the equations

$$X_a^{27} = X_0^{27-a} X_{27}^a$$

hold on C; both endpoints vanishing would force every coordinate to vanish at a projective point.

After inverting t^{13} in R, we recover $t = t^{14}/t^{13}$ and $t^{-1} = t^{38}/(t^{13})^3$. Thus $R[(t^{13})^{-1}] = k[t, t^{-1}]$. The locus $t^{13} = 0$ is just the origin: powers of each other generator vanish there by their monomial relations. Hence the first chart is smooth off the origin, while the entire second chart is smooth. The origin is singular, since its local ring has eight independent minimal generators of its maximal ideal and dimension one. The map is birational and the two chart normalizations glue to $\mathbb{P}^1$. The normalization quotient is supported only at the origin, where its basis consists of t^g for $g \in \mathcal{G}$. Its length is 17. The normalization exact sequence, with $\chi(\mathcal{O}_{\mathbb{P}^1}) = 1$, gives $\chi(\mathcal{O}_C) = 1 - 17$ and arithmetic genus 17. □

In particular, C has no embedded points. Its local rings are Cohen–Macaulay: a one-dimensional local domain has depth one, and the generic local ring is a field.

3 The full tangent space at a canonical ideal

Write $\widehat{R} = k[[t^S]]$ for the completed local ring at the origin, and $\overline{\widehat{R}} = k[[t]]$ for its normalization. A normalized canonical fractional ideal of $\widehat{R}$ is

$$K = \widehat{R}\langle 1, t, t^3, t^5, t^6\rangle. \tag{5}$$

For clarity, the standard semigroup formula identifies its exponent set with $\{n \in \mathbb{Z} : 25 - n \notin S\}$; see the canonical-module description in [3, Section 6] and [2, Section 2]. Equivalently, the pseudo-Frobenius numbers are $19, 20, 22, 24, 25$, and subtracting them from 25 gives the five generators in (5). Both descriptions work over any field. The finite checks below, together with the conductor tail, also explicitly verify these lists.

The additional exponents of K over $\widehat{R}$ are

$$\{1, 3, 5, 6, 19, 20, 22, 24\}; \qquad \ell(K/\widehat{R}) = 8.$$

We need an inverse fractional ideal, not an unproved identification of a colon with a trace ideal.

Lemma 3.1. *One has $\widehat{R} : K = t^{26}k[[t]]$. The ideal $\widehat{I} = t^{26}K$ has colength 18.*

Proof. Since $1 \in K$, any inverse multiplier lies in $\widehat{R}$. Every exponent at least 26 multiplies K into $\widehat{R}$. Each remaining possible exponent s fails one of the following monomial tests:

s	0	13	14	15	16	17	18	21	23
addend from K	1	6	5	5	3	3	1	1	1
sum outside S	1	19	19	20	19	20	19	22	24

For a power series, multiplication by the specified monomial shifts each coefficient to a different exponent, so no forbidden coefficient can be canceled. This proves the full colon equality.

Multiplying (5) by t^{26} gives precisely the generators of I. Its quotient monomials have exponents

$$\mathcal{B} = \{0, 13, 14, 15, 16, 17, 18, 21, 23, 28, 30, 33, 34, 35, 36, 37, 38, 51\}. \tag{6}$$

To check completeness, membership is $n - 26 \in \mathrm{val}(K)$; below 52 this gives exactly (6), and every exponent at least 52 is in I. Thus the colength is 18, both in R and in its completed local ring. The support is the origin. □

The ideal becomes the unit ideal off the origin and hence glues with $\mathcal{O}_C$ on the second chart to define $Z \subset C$.

Proposition 3.2. *Multiplication induces an isomorphism*

$$T_{[Z]} \operatorname{Hilb}^{18}(C) \simeq \operatorname{Hom}_{\widehat{R}}(\widehat{I}, \widehat{R}/\widehat{I}) \simeq k[[t]]/\widehat{R}.$$

In particular this is the entire Hilbert tangent space and has dimension 17*.*

Proof. The Hilbert tangent space is $\operatorname{Hom}_C(\mathcal{I}_Z, \mathcal{O}_Z)$. Its target has finite support at the origin. Localization and then completion identify this with the displayed module Hom: the source is finitely presented, completion is flat, and finite-length target modules are unchanged by completion.

A canonical module of a Cohen–Macaulay complete local ring is semidualizing. More explicitly, the normalized dualizing complex is the canonical module shifted by the dimension, and its derived self-Hom is the ring in degree zero. Taking cohomology gives

$$\operatorname{Hom}_{\widehat{R}}(K, K) = \widehat{R}, \qquad \operatorname{Ext}^j_{\widehat{R}}(K, K) = 0 \quad (j > 0).$$

These follow from the definition of a dualizing complex and its Cohen–Macaulay concentration; see [5, Definition 47.15.1 and Lemma 47.20.2]. They hold as well for $\widehat{I} \simeq K$. Applying $\operatorname{Hom}_{\widehat{R}}(\widehat{I}, -)$ to the ideal exact sequence gives

$$0 \longrightarrow \widehat{R} \longrightarrow (\widehat{R} : \widehat{I}) \longrightarrow \operatorname{Hom}_{\widehat{R}}(\widehat{I}, \widehat{R}/\widehat{I}) \longrightarrow 0.$$

Here maps between rank-one fractional ideals are multiplication by fractions: this follows by tensoring a map with the fraction field, where both modules have dimension one. Lemma 3.1 yields $\widehat{R} : \widehat{I} = t^{-26}(\widehat{R} : K) = k[[t]]$. The first map is inclusion of multiplier sets. The quotient therefore has the seventeen gap monomials (4) as a basis. □

Remark 3.3. The self-Ext vanishing above is a local module statement. It is not the generally false assertion $\operatorname{Ext}^1_C(\omega_C, \omega_C) = 0$ on the projective curve; the latter contains the global Jacobian directions. Reduction to a finite-support target is essential. No condition fixing support was imposed on the deformations used to compute the full Hilbert tangent space.

Already Proposition 3.2 answers the existential AIM question. At a closed point of a finite-type scheme, the local dimension is at most the tangent dimension. Every irreducible component through $[Z]$ therefore has dimension at most $17 < 18$. This argument concerns actual global components, not merely a small-dimensional punctual stratum. The next section supplies the matching lower bound and rationality.

4 A flat unit family and an open affine space

Put $B = k[t]/(t^{26})$, and let $A \subset B$ be the nine-dimensional subalgebra with the exponents in (3). Let $E \subset B$ be the seventeen-dimensional A-submodule obtained from K, with exponent set

$$\mathcal{E} = \{0, 1, 3, 5, 6, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24\}.$$

For any commutative k-algebra T, and parameters $(c_g)_{g \in \mathcal{G}}$, the element

$$u = 1 + \sum_{g \in \mathcal{G}} c_g t^g \in B_T^\times$$

is a unit, because its positive-degree part is nilpotent. The high degree block of $V = R/(t^{52}k[t])$ is $t^{26}B$. Define

$$J_u = \pi_T^{-1}(t^{26}uE_T) \subset R \otimes_k T, \qquad \pi_T : R \otimes_k T \longrightarrow V_T. \tag{7}$$

The tail here is the full ideal $t^{52}k[t]$, not $t^{52}R$.

Lemma 4.1. *Equation* (7) *defines a flat family of length-*18 *subschemes supported at the origin, with* $J_1 = I$.

Proof. Multiplication on the high block factors through A_T. Commutativity and $A_TE_T \subset E_T$ imply $A_T(uE_T) \subset uE_T$, so (7) is an ideal. Multiplication by u is an invertible T-linear map of B_T. The coordinate subspace E_T is a free direct summand of rank 17, so B_T/uE_T is free of rank 9. The quotient by J_u is the direct sum, as a T-module, of this rank-9 high quotient and the unaffected rank-9 low block. It is free of rank 18. Its support lies at the origin because the full tail is in the ideal. For $u = 1$ the exponent list is exactly that of I. □

Thus we have a morphism $\Phi : \mathbb{A}^{17} \to \mathrm{Hilb}^{18}(C)$. All quotients lie in the affine chart; the corresponding locus is open in the projective-curve Hilbert scheme. It would be incorrect to define the family by unqualified polynomial multiplication uI: a polynomial with constant term one may vanish elsewhere. The truncated preimage in (7) avoids such unintended support.

Lemma 4.2. *For every* k*-algebra* T*, the stabilizer of* E_T *in* $B_T^\times$ *is* $A_T^\times$*. Every coset has a unique representative with constant coefficient one and all positive nongap coefficients zero. Consequently* Φ *is a monomorphism.*

Proof. If $v = \sum_{j=0}^{25} v_j t^j$ preserves E_T, then $v_j = 0$ whenever some $e \in \mathcal{E}$ satisfies $j + e < 26$ and $j + e \notin \mathcal{E}$, by inspecting the coefficient of t^{j+e} in vt^e. For $j \notin \mathcal{E}$, use $e = 0$. For the remaining gaps use

j	1	3	5	6	19	20	22	24
e	1	1	3	1	6	5	3	1
$j+e$	2	4	8	7	25	25	25	25

to kill every gap coefficient. Thus $v \in A_T^\times$. Conversely every unit of A_T and its inverse preserve E_T. The argument is coefficientwise over arbitrary T, including nonreduced rings, so it identifies the scheme-theoretic stabilizer.

For the normal form, first divide a unit by its constant coefficient. Multiply it by $1+\sum_s h_s t^s$, where s runs over the eight positive nongaps below 26. In increasing order of s, choose h_s to kill the coefficient at t^s. The new unknown enters with coefficient one and earlier coefficients are unaffected. This also proves uniqueness by induction. All operations are polynomial after inverting the original constant coefficient. Thus these normal forms represent the quotient $B^\times/A^\times$ as $\mathbb{A}^{17}$ over k, not only its set of closed points. If $J_u = J_v$, their high blocks give $uE_T = vE_T$, so $v^{-1}u \in A_T^\times$ and uniqueness forces $u = v$. The represented Hilbert functor therefore gives a monomorphism. □

Proof of Theorem 1.1. The morphism Φ has zero-dimensional geometric fibers, so the closure of its image is irreducible of dimension 17. It contains $[I]$. The full tangent dimension there is 17, so local dimension and embedding dimension agree: the local ring is regular.

The same holds at every closed point in this family. After completing at the origin, $J_u = u\widehat{I}$, because a lift of the truncated unit is a unit of $k[[t]]$ and $\widehat{I}$ contains its conductor tail. Thus J_u is again a canonical module, its endomorphism ring is $\widehat{R}$, and

$$\widehat{R} : J_u = u^{-1}k[[t]] = k[[t]].$$

The argument of Proposition 3.2 again gives full tangent dimension 17. Each family point lies on the same 17-dimensional image closure, hence the Hilbert scheme is regular there. Since the algebraically closed field is perfect, these points are smooth. Regularity also implies that the image closure is the unique irreducible component through each such point.

A monomorphism induces an injective map on tangent spaces. Both source and target are smooth of dimension 17 at these points, so the differential is an isomorphism and Φ is etale there. The non-etale locus in $\mathbb{A}^{17}$ is closed and, if nonempty, would contain a closed point. It is therefore empty. An etale monomorphism is an open immersion. Its image is consequently an open $\mathbb{A}^{17}$ in the Hilbert scheme, dense in the rational component just found. □

5 Components in all lengths at least eighteen

We now assume characteristic zero and prove Corollary 1.2 using Kass's theorem. This is independent of the explicit unit-orbit lower bound. For rank-one torsion-free sheaves on C, degree means $\deg F = \chi(F) - \chi(\mathcal{O}_C)$.

Glue the fractional module $R\langle 1, t, t^3, t^5, t^6\rangle$ on the first chart to $\mathcal{O}_C$ on the smooth second chart, using their common restriction to $\operatorname{Spec} k[t, t^{-1}]$, and call the resulting sheaf F. The canonical excess list and colon witnesses give exact sequences

$$0 \to \mathcal{O}_C \to F \to Q \to 0, \quad \ell Q = 8, \qquad 0 \to F^\vee \to \mathcal{O}_C \to Q' \to 0, \quad \ell Q' = 9.$$

Hence $\deg F = 8$ and $\deg F^\vee = -9$.

Both F and ω_C are locally canonical at the sole singular point and are invertible elsewhere. Therefore $L = \mathcal{H}om(F, \omega_C)$ is a line bundle and evaluation is an isomorphism $F \otimes L \simeq \omega_C$. Locally at the singular point this follows from canonical self-Hom; elsewhere it is the usual evaluation between line bundles. Since $g(C) = 17$,

$$\deg \omega_C = 2g - 2 = 32, \quad \deg L = 24, \quad \omega_C^\vee = F^\vee \otimes L^{-1}, \quad \deg \omega_C^\vee = -33.$$

Only line-bundle twisting has been used for degree additivity. No additive rule for arbitrary tensor products of noninvertible rank-one torsion-free sheaves is being assumed.

The singularity is non-Gorenstein: the canonical module has five minimal generators in (5). In Kass's notation [4, Definition 2.3],

$$d_0 = 1 - \deg \omega_C^\vee = 34, \qquad n_d = d + 2g - 2 + \deg \omega_C^\vee = d - 1.$$

His Theorem 2.7(3) applies to an integral projective curve over an algebraically closed field of characteristic zero with a unique non-Gorenstein singularity, and yields an irreducible component of dimension n_d for $d \geq d_0 - g + 1$. These hypotheses are now all verified; the threshold is $34 - 17 + 1 = 18$. This proves Corollary 1.2.

More precisely, Kass's Lemmas 2.4 and 2.6 exhibit an open part as a projective-space bundle with fiber $\mathbb{P}^{d-18}$ over a 17-dimensional locus. Its closure is a component of the full Hilbert scheme, not just a punctual support locus. No finite representation type hypothesis is required in Theorem 2.7; that hypothesis occurs in his separate component-enumeration theorem.

6 Independent finite tangent certificates

The supplemental standard-library programs compute $\operatorname{Hom}_R(I, R/I)$ without canonical modules or Kass's theorem. There are five generator images, each with the eighteen coefficients in

(6), hence 90 unknowns. Write $(a_1, \ldots, a_5) = (26, 27, 29, 31, 32)$. For each common degree $w \in (a_i + S) \cap (a_j + S)$ there is a homogeneous syzygy

$$t^{w-a_i} e_i - t^{w-a_j} e_j. \tag{8}$$

These span the entire relation module: in a fixed degree all nonzero basis monomials map to the same t^w, so their kernel is spanned by differences. There are no ungraded relations left over, since every polynomial relation splits into homogeneous ones.

The first implementation uses all pairs with

$$w < \max(a_i, a_j) + 26 + 13.$$

Every larger common degree can be decreased by 13, since both remaining exponents are at least 26 and belong to S. Thus (8) is t^{13} times a lower-degree relation; iteration terminates within the enumerated range. This is a proof of a complete bound, not a numerical cutoff guess.

The independently written implementation instead takes a basis of the relation space in each degree $26 \leq w < 84$. At every degree $w \geq 84$, every multiplier has exponent at least $84 - 32 = 52$ and belongs to I, so it annihilates R/I. Those higher relations therefore impose no further equations. The two complete systems give

Generation method	Unknowns	Scalar equations	Rank
All pairs, reduction by t^{13}	90	683	73
Graded basis, annihilator cutoff	90	361	73

Every scalar equation is $x_i = x_j$ or $x_i = 0$. Regard a zero constraint as an edge to an extra zero vertex. A recorded spanning forest has 73 edges, leaving 17 freely assignable components. This proves the rank over every field, rather than extrapolating from finitely many prime characteristics. The free components have degree shifts exactly (4); the corresponding explicit maps are $f \mapsto t^g f \pmod{I}$ for $g \in \mathcal{G}$.

The supplement includes both programs, complete equation data, the forest, and instructions for bit-for-bit regeneration. Rational elimination and ranks modulo $2, 3, 5, 7, 101$ provide additional cross-checks, but the graph and complete syzygy bounds are the characteristic-independent mathematical certificate.

7 Scope, attribution and verification

The result supplies an affirmative example for the original existence question. It does not classify all components for this curve, determine the least possible length, or establish that length 18 is minimal. The relevant semigroup and canonical data are positively identified in [3, 2]. Our global consequence is explicitly obtained from [4]. A targeted primary-source search did not locate this particular Hilbert-scheme application in those works or the nearby literature checked; that non-discovery does not certify absolute priority or exclude other prior answers.

The proof, curve conventions, degree computation and two exact syzygy systems were checked in separate internal AI-assisted research lanes. The main proof and the unit-family argument were read adversarially, including the distinction between local and global self-Ext, the necessity of the full tangent space, and the truncated definition of the family. No external human peer review or formal proof-assistant verification has been performed. The manuscript is an unrefereed preprint; correctness remains open to scrutiny.

AI systems assisted with literature retrieval, mathematical development, independent internal checking, exact verification code, and manuscript preparation. These internal checks are not represented as human refereeing or a formal proof. Established results and the original problem source are cited explicitly.

References

[1] American Institute of Mathematics. Components of Hilbert schemes: Problems. AIM workshop, July 19–23, 2010, 2010. Recorded by Izzet Coskun and edited by Li Li. Problem 20, p. 3. https://aimath.org/WWN/hilbertschemes/hilbertschemes.pdf.

[2] Jürgen Herzog and Shinya Kumashiro. Upper bound on the colength of the trace of the canonical module in dimension one. *Archiv der Mathematik*, 119(3):237–246, 2022. Inspected version: arXiv:2201.12508v1, Lemma 2.2 and Example 2.5, p. 4. https://arxiv.org/abs/2201.12508v1.

[3] Jürgen Herzog, Shinya Kumashiro, and Dumitru I. Stamate. The tiny trace ideals of the canonical modules in Cohen–Macaulay rings of dimension one. *Journal of Algebra*, 619:626–642, 2023. Inspected version: arXiv:2106.09404v1 (2021), Example 5.4(iii), p. 10. https://arxiv.org/abs/2106.09404v1.

[4] Jesse Leo Kass. An explicit non-smoothable component of the compactified Jacobian. *Journal of Algebra*, 370:326–343, 2012. Inspected version: arXiv:1108.2706v2, Definition 2.3 and Theorem 2.7(3), pp. 5–6. https://arxiv.org/abs/1108.2706v2.

[5] The Stacks Project Authors. The Stacks project: Dualizing complexes and Cohen–Macaulay rings, 2026. Definition 47.15.1, tag 0A7B, and Section 47.20, Lemma 47.20.2. Accessed September 5, 2026. https://stacks.math.columbia.edu/tag/0A7B; https://stacks.math.columbia.edu/tag/0DW4.