A Diagonal-Jet Criterion for Normalized Gaussian Analytic Covariances

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Abstract

Let R be a twice continuously differentiable, positive semidefinite complex kernel with unit diagonal on a connected plane domain. We give a necessary and sufficient condition for R to be the normalized covariance of a centered proper Gaussian analytic function with positive variance everywhere. A nonnegative residual formed from diagonal derivatives must vanish, and a real one-form determined by the first diagonal derivative must be exact. On a simply connected domain the second condition is a local closedness test. A sharp Gram inequality propagates the diagonal residual condition without division by any off-diagonal kernel value. The covariance is reconstructed up to a positive constant. A polynomial example has a zero in every base section, and an annular example separates local conditions from the global period obstruction. This elementary note concerns the full complex normalized covariance, not arbitrary zero-process correlations. It uses classical positive-kernel and Gaussian-series methods and makes no absolute priority claim.

1 The normalized realization problem

Let $D \subset \mathbb{C}$ be a connected open set. A centered proper Gaussian analytic function (GAF) on D is a random holomorphic function whose finite-dimensional distributions are centered complex Gaussian and whose pseudo-covariance $\mathbb{E}[f(z)f(w)]$ vanishes. Write

$$K(z,w) = \mathbb{E}[f(z)\overline{f(w)}].$$

Assuming $K(z,z) > 0$ everywhere, its normalized complex covariance is

$$R(z,w) = \frac{K(z,w)}{\sqrt{K(z,z)K(w,w)}}. \tag{1}$$

We ask when a prescribed full complex kernel R has this form. Both its modulus and phase are part of the data. All positive semidefinite (PSD) kernel assumptions mean positivity of *every* finite Gram matrix, not just selected two-point or numerical tests.

An AIM workshop question attributed to Steven Evans asks for a correlation-function characterization of Gaussian entire functions [1]. The source does not define its correlation data. Ordinary field covariance, normalized covariance, and correlations of zeros are different problems. We address (1) explicitly, and do not claim to resolve other readings of that source question.

The frozen research aid attached to record AIM-ANALYSIS-0164 in [4] gives a normalized criterion on $\mathbb{C}$ under the extra assumption that some section $R(\cdot, z_0)$ never vanishes. Its method divides by that section. The criterion below removes this assumption. Section 5 shows that no change of base

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point can always make the assumption hold. It also explains why local tests alone fail on nonsimply connected domains.

Our proof uses standard Gram and Gaussian-series arguments. Related positivity-driven propagation of regularity appears in Buescu–Paixão–Oliveira [3, Theorem 3.13]: a PSD kernel sesquiholomorphic near the entire diagonal is sesquiholomorphic globally. The normalized kernel in (1) is generally not sesquiholomorphic. The additional task here is to recover its positive real amplitude, including the global period condition. The resulting criterion is an elementary package of classical ideas; priority is not asserted.

2 Statement and a sharp residual inequality

Use $\partial_z = (\partial_x - i\partial_y)/2$ and $\partial_{\bar z} = (\partial_x + i\partial_y)/2$. For a C^2 kernel R on $D \times D$, define

$$\begin{aligned} A(z) &= \partial_{\bar z} R(z,w)\big|_{w=z}\,, & & \\ B(z) &= \partial_{\bar z}\partial_w R(z,w)\big|_{w=z}\,, & E(z) &= \partial_z \partial_{\bar w} R(z,w)\big|_{w=z}\,, \\ q_-(z) &= B(z) - |A(z)|^2, & q_+(z) &= E(z) - |A(z)|^2. \end{aligned} \tag{2}$$

Derivatives in (2) are taken before setting $w = z$. Define the real one-form

$$\omega = -2\,\mathrm{Re}(\overline{A(z)}\,\mathrm{d}z). \tag{3}$$

Theorem 1. *Let $R : D \times D \to \mathbb{C}$ be C^2 in its four real variables, Hermitian PSD, and satisfy $R(z,z) = 1$. It is the normalized covariance of a centered proper GAF with positive variance everywhere if and only if*

$$q_- \equiv 0 \quad \text{and} \quad \omega \text{ is exact on } D. \tag{4}$$

If $\mathrm{d}h = \omega$ for real h, the reconstructing covariance is

$$K(z,w) = e^{h(z)+h(w)} R(z,w). \tag{5}$$

All covariances realizing the same R are positive constant multiples of this one. On a simply connected domain, exactness in (4) is equivalent to

$$\mathrm{Im}(\partial_z A) = 0. \tag{6}$$

Every realizable R is in fact real analytic.

Thus adding C^2 regularity to the conditions gives a characterization even if no regularity is initially assumed of R. The case $D = \mathbb{C}$ is the entire-function version.

Lemma 2. *For every kernel in the hypotheses of Theorem 1, $q_-, q_+ \geq 0$, and*

$$\big|\partial_{\bar z} R(z,w) - A(z) R(z,w)\big|^2 \leq q_-(z)\big(1 - |R(z,w)|^2\big). \tag{7}$$

The constant one is sharp.

Proof. The Gram construction yields a Hilbert space and a map $u : D \to H$ with $R(z,w) = \langle u(z), u(w)\rangle$ and $\|u(z)\| = 1$. Inner products are linear in the first argument. The map u is norm-continuous by the formula for the squared norm of an increment.

We justify norm differentiability at the stated regularity. For a real coordinate direction e, the inner product of difference quotients is

$$\begin{aligned}&\left\langle \frac{u(z+te)-u(z)}{t}, \frac{u(w+se)-u(w)}{s} \right\rangle \\ &\qquad = \frac{R(z+te,w+se)-R(z+te,w)-R(z,w+se)+R(z,w)}{ts}.\end{aligned}$$

The right side is the double integral average of a continuous mixed second derivative. At $w = z$, the norm squared of the difference between the quotients with steps t, s tends to zero, since their squared norms and mutual inner product have the same limit. Completeness gives a norm partial derivative. The mixed derivative formula at two base points proves its norm continuity. Applying this to both real directions gives $u \in C^1(D; H)$ and the corresponding Wirtinger Gram identities. No second derivative of u is needed.

Put $v = \partial_{\bar z} u$. Then

$$A = \langle v, u\rangle, \qquad B = \|v\|^2.$$

The vector $v - Au$ is perpendicular to u and has squared norm q_-. Also, differentiating $R(z,z) = 1$ gives

$$C(z) := \partial_z R(z,w)|_{w=z} = -\overline{A(z)}. \tag{8}$$

The analogous residual for $\partial_z u$, whose squared norm before projection is E, gives $q_+ \geq 0$.

Project $u(w)$ off $u(z)$. Its residual has squared norm $1 - |R(z,w)|^2$. The inner product of $v - Au$ with this residual is $\partial_{\bar z} R(z,w) - A(z)R(z,w)$. Cauchy–Schwarz proves (7). For sharpness take $u(z) = (\cos x, \sin x)$, $x = \operatorname{Re} z$. Then $R(z,w) = \cos(x - \operatorname{Re} w)$, $A = 0$, $q_- = 1/4$, and both sides of (7) equal $\frac{1}{4}\sin^2(x - \operatorname{Re} w)$. □

In particular, $q_- = 0$ implies the global equation

$$\partial_{\bar z} R(z,w) = A(z)R(z,w) \qquad (z, w \in D). \tag{9}$$

This implication remains valid at every off-diagonal zero of R.

3 Proof of the realization criterion

Write $A = a + ib$. The one-form (3) is $-2a\,\mathrm{d}x - 2b\,\mathrm{d}y$. A real potential satisfies

$$\partial_{\bar z} h = -A, \tag{10}$$

and conversely (10) is exactly $\mathrm{d}h = \omega$. Since $A \in C^1$, the potential is C^2. Moreover,

$$\mathrm{d}\omega = -2(b_x - a_y)\,\mathrm{d}x \wedge \mathrm{d}y = -4\operatorname{Im}(\partial_z A)\,\mathrm{d}x \wedge \mathrm{d}y. \tag{11}$$

On a simply connected domain, closedness suffices. On a general domain it must be supplemented by zero integrals over all closed piecewise smooth curves. One may then reconstruct h by integrating ω from a fixed base point.

Sufficiency and the classical Gaussian series

Assume (4) and choose h as above. Positive diagonal congruence of every Gram matrix shows that (5) is PSD. Equations (9) and (10) give $\partial_{\bar z} K = 0$. Hermitian symmetry gives $\partial_w K = 0$, so K is sesquiholomorphic, with $K(z,z) = e^{2h(z)} > 0$.

We recall the standard realization, including the convergence step [2, Section 2.2, Lemma 2.2.3]. Apply the Gram construction to K and denote its feature map by $k(z)$. Restrict the Hilbert space to the closed feature span. It is separable: features at a countable dense subset of D span densely by continuity. For each vector b, $\langle k(z), b\rangle$ is holomorphic. Indeed this holds for a finite combination of features by sesquiholomorphy, and approximation in norm is locally uniform because $\|k(z)\|^2 = K(z,z)$ is locally bounded.

For an orthonormal basis (b_j), put $e_j(z) = \langle k(z), b_j\rangle$. Parseval gives

$$K(z,w) = \sum_j e_j(z)\overline{e_j(w)}, \qquad K(z,z) = \sum_j |e_j(z)|^2.$$

The diagonal series converges uniformly on each compact set by Dini's theorem. Let ξ_j be independent proper standard complex Gaussians. For every relatively compact open U with $\overline{U} \subset D$,

$$\sum_j \mathbb{E}\|\xi_j e_j\|^2_{A^2(U)} = \int_U K(z,z)\,\mathrm{d}A(z) < \infty.$$

The independent Hilbert-space series therefore converges almost surely in the Bergman space $A^2(U)$. Equivalently its partial sums form a bounded square-integrable Hilbert-valued martingale. The analytic mean-value inequality converts this convergence into uniform convergence on compact subsets of U. A countable exhaustion yields a.s. local uniform convergence of $f = \sum_j \xi_j e_j$ on D. Evaluation also converges in L^2, giving covariance K, zero pseudo-covariance and the required Gaussian finite-dimensional laws. Formula (1) is exactly the prescribed R.

Necessity, regularity and uniqueness

Suppose f realizes R with covariance K. We first recall why K is continuous and sesquiholomorphic without assuming a moment bound for a random supremum. If $z_n \to z$, path continuity gives convergence in probability of $f(z_n) - f(z)$. Centered proper Gaussian increments then converge in L^2: a variable of variance $s > 0$ has tail $\mathbb{P}(|X| > t) = e^{-t^2/s}$, which forces $s \to 0$ in this situation. Consequently the covariance is continuous and its diagonal is bounded on compact sets. Cauchy–Schwarz justifies Fubini along a triangle, so the sample-path Cauchy theorem and Morera's theorem prove holomorphy of $K(\cdot, w)$. Symmetry gives antiholomorphy in w. Separate holomorphy, after conjugating the second coordinate, gives joint analyticity.

Set $h(z) = \frac{1}{2}\log K(z,z)$. Differentiating $R = e^{-h(z)-h(w)}K$ gives (9) with $A = -\partial_{\bar z} h$. A further ∂_w derivative and diagonal restriction give $B = A\overline{A}$, hence $q_- = 0$. Also $\mathrm{d}h = \omega$, proving necessity. This argument shows that h and R are real analytic.

Two real logarithmic amplitudes realizing the same full complex R have difference annihilated by $\bar\partial$. A real holomorphic function on connected D is constant. Their covariances thus differ by a positive constant, completing the proof of Theorem 1. The restriction to proper centered GAFs ensures that covariance determines the Gaussian law; no analogous uniqueness is asserted with an unspecified nonzero pseudo-covariance.

4 The usual zero intensity in diagonal-jet notation

Differentiating the diagonal restrictions in (2) and (8) gives

$$\partial_z A = (R_{z\bar z} + R_{\bar z w})|_{w=z}, \qquad \partial_{\bar z} C = (R_{z\bar z} + R_{z\bar w})|_{w=z}.$$

Since $C = -\overline{A}$, subtraction yields

$$2\,\mathrm{Re}(\partial_z A) = B - E = q_- - q_+. \tag{12}$$

When Theorem 1 applies, closedness and $q_- = 0$ give $\partial_z A = -q_+/2$. The classical Edelman–Kostlan formula [2, Equation (2.4.8)], with $\Delta = 4\partial_z\partial_{\bar z}$, therefore becomes

$$\rho_1(z) = \frac{1}{4\pi}\Delta \log K(z,z) = -\frac{2}{\pi}\partial_z A(z) = \frac{q_+(z)}{\pi}. \tag{13}$$

This is a restatement of the known formula, not a new zero-intensity theorem. Positive variance excludes deterministic zero atoms. There is no claim about realization of arbitrary prescribed higher zero correlation functions.

5 Examples and indispensable hypotheses

Fock covariance

For $R(z,w) = e^{z\bar w - (|z|^2+|w|^2)/2}$ one has $A = -z/2$, $q_- = 0$, $q_+ = 1$ and $h = |z|^2/2$, normalized at zero. Reconstruction gives $K = e^{z\bar w}$, and (13) gives the standard intensity $1/\pi$.

A zero in every base section

Let ξ_0, ξ_1, ξ_2 be independent proper standard complex Gaussians and take

$$f(z) = \xi_0 + \xi_1 z + \xi_2(z^2+1), \qquad K(z,w) = 1 + z\bar w + (z^2+1)(\bar w^2 + 1).$$

The variance $1 + |z|^2 + |z^2+1|^2$ is positive everywhere. For each fixed w, the coefficients of z and z^2 in $K(z,w)$ are $\bar w$ and $\bar w^2 + 1$, which cannot both vanish. Thus every section is a nonconstant polynomial and has a complex zero. The positive normalization preserves its zeros. At $w = \pm i$ the section is linear rather than quadratic. Theorem 1 covers this normalized covariance, although no global zero-free base section exists.

Exactness does not imply zero residual

The real Gaussian kernel $R(z,w) = e^{-|z-w|^2}$ is PSD on $\mathbb{C}$. Indeed, after positive diagonal factors it is $e^{z\bar w}e^{\bar z w}$, a product of two PSD kernels. Its jets are $A = 0$ and $q_- = q_+ = 1$. Its one-form is exact, but it fails the first condition in (4).

Zero residual does not imply closedness

The rank-one kernel $R(z,w) = e^{i(|z|^2-|w|^2)}$ is PSD and unit-diagonal. It has $A = iz$ and $q_- = q_+ = 0$, but $\partial_z A = i$, so its one-form is not closed. This example is inherited from the research aid [4]; it is not presented as a new counterexample.

Closedness does not eliminate a period

On $D = \{1 < |z| < 2\}$, fix $\alpha \in \mathbb{R} \setminus \{0\}$ and set

$$R_\alpha(z,w) = e^{i\alpha(\log|z| - \log|w|)}.$$

This is a smooth rank-one PSD kernel with unit diagonal. Directly,

$$A = \frac{i\alpha}{2\bar z}, \qquad q_- = 0, \qquad \partial_z A = 0, \qquad \omega = -\alpha\, \mathrm{d}\arg z.$$

The one-form is closed, but its integral on a positively oriented circle is $-2\pi\alpha \neq 0$. Hence there is no global realization. Locally, choose a branch of $\log z$ and put $g = z^{i\alpha}$. Then ξg has normalized covariance R_α. The example therefore separates local realization from global realization. Nonzero integer α does not repair the obstruction: the amplitude monodromy is $e^{-2\pi\alpha}$, so periods must vanish exactly, not merely modulo 2π.

6 Scope, reproducibility and status

The accompanying standard-library checker uses exact rational complex arithmetic and a second-order four-real-variable Taylor algebra. It checks 1,381 diagonal instances across six kernel families and 3,072 finite Gram configurations, including degenerate and sharp cases. In total, 13,295 assertions pass in isolated runs on Python 3.9.6 and 3.13.5, with matching results apart from runtime metadata. A mutation test rejects an erroneous extra term in a diagonal chain rule. These are finite regression checks, not a proof of universal positivity or a substitute for the analytic argument. Two runtime executions are not independent mathematical review.

The theorem settles the specified normalized-complex-covariance problem, without a zero-free section assumption. It does not establish what was intended by the undefined correlation data in the original AIM question. The Gaussian-series realization, the positive-kernel framework, and (13) are classical and credited above. A bounded literature search did not identify this precise diagonal-jet-and-period formulation in the inspected sources, but it may be implicit or folklore; no absolute novelty or priority is claimed.

This is an unrefereed preprint. AI assistance was used in research, derivation, drafting and reproducibility checks; responsibility for the manuscript rests with the author. No external mathematical review or proof-assistant verification is represented as completed.

References

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[3] J. Buescu, A. Paixão and C. Oliveira, Propagation of regularity and positive definiteness: a constructive approach, *Z. Anal. Anwend.* **37** (2018), 1–24, doi:10.4171/ZAA/1599.

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