

Eventual Scalar Recurrences for Constrained Orthogonal Polynomials

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12 October 2026

Abstract

Let S be a finite-codimensional subspace of $\mathbb{R}[x]$ with a positive definite moment inner product, and let q_n be its degree-ordered monic orthogonal basis. For a fixed nonzero polynomial a of degree d , we prove that an eventual scalar band recurrence for aq_n exists if and only if multiplication by a preserves $S + S^\perp$, where the orthogonal complement is taken inside the polynomial space. The width can then be chosen to be d . Some nonconstant multiplier works exactly when this sum contains a nonzero polynomial ideal. For the inner product on $[0, 1]$ and the single constraint $\int_2^3 p(x) dx = 0$, every nonconstant multiplier instead sends infinitely many q_n outside S . A mean-zero Legendre example shows why stabilizing S itself is not necessary for a tail recurrence. These elementary results refine an established stabilizer approach; they do not classify differential, coupled, or compressed-operator recurrences, or settle the broader AIM problem. The note is unrefereed and makes no absolute priority claim.

1 Setting and relation to prior work

Write $V = \mathbb{R}[x]$ and $V_n = \{p \in V : \deg p \leq n\}$, with $V_{-1} = \{0\}$. Let μ be a positive measure on $\mathbb{R}$ with infinite support and all moments finite. Thus

$$\langle p, q \rangle = \int p(x)q(x) d\mu(x)$$

is positive definite and multiplication by any real polynomial is symmetric. Let $S \subset V$ have finite codimension k , without any continuity assumption on the linear functionals defining it. Let I be the set of degrees attained by nonzero elements of S . For $n \in I$, denote by q_n the unique monic element of S of degree n orthogonal to $S \cap V_{n-1}$.

For completeness, if C_n is the matrix of k independent defining constraints evaluated on $1, x, \dots, x^n$, then

$$\dim(S \cap V_n) = n + 1 - \text{rank } C_n.$$

A degree is missing precisely when this rank increases. The rank eventually equals k , so there are exactly k missing degrees. Finite-dimensional orthogonal projection constructs each q_n ; degree reduction shows that these polynomials form an orthogonal algebraic basis of S . These are standard rank and Gram constructions.

The AIM workshop report asks about recurrence relations for orthogonal polynomials subject to linear constraints and suggests compressed multiplication as one approach [1, Problem 6.3, p. 4].

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Normand already explains the generic failure of fixed-length recurrences in the block-orthogonal setting [2, §3.3.5]. Stabilizer rings and polynomial-multiplier band recurrences are explicit in the work of Gómez-Ullate, Kasman, Kuijlaars and Milson [3, Definition 5.1 and Remark 5.2]. Recent Geronimus constructions also treat special constrained and lacunary families [4].

Here we distinguish an eventual recurrence from invariance of the original constrained space. Our result concerns a single fixed polynomial multiplier and scalar coefficients. It does not exclude relations with derivatives, several degree-dependent polynomial multipliers, extra functions, or compressed operators. The real moment setting can be complexified for real constraints, but arbitrary non-Hankel inner products and the wider orthogonal-basis and Padé questions are not treated.

2 The polynomial orthogonal complement

Define

$$T = \{t \in V : \langle t, s \rangle = 0 \text{ for every } s \in S\}, \quad R = S + T.$$

These are algebraic polynomial spaces, not Hilbert-space closures. Positivity gives $S \cap T = \{0\}$, so T injects into V/S . In particular, $\dim T \leq k$ and $R = S \oplus T$ has finite codimension. Set $D = -1$ if $T = \{0\}$; otherwise let D be the largest degree of a member of a basis of T . Every polynomial in T then has degree at most D .

Lemma 1. *A polynomial $v \in V$ satisfies $\langle v, q_n \rangle = 0$ for all sufficiently large $n \in I$ if and only if $v \in R$.*

Proof. If $v = s + t$ with $s \in S$ and $t \in T$, use the finite basis expansion of s . Conversely, under the stated vanishing condition the sum

$$s = \sum_{n \in I} \frac{\langle v, q_n \rangle}{\langle q_n, q_n \rangle} q_n$$

has only finitely many nonzero terms. Then $v - s$ is orthogonal to every q_n , hence to their algebraic span S , and belongs to T . No completeness or infinite-series argument is involved. $\square$

For a fixed nonzero $a \in \mathbb{R}[x]$, an *eventual scalar band recurrence* means that there are integers $N, w \geq 0$ such that the polynomial identities

$$a(x)q_n(x) = \sum_{\substack{m \in I \\ |m-n| \leq w}} c_{n,m} q_m(x) \quad (n \in I, n \geq N) \quad (1)$$

hold with real scalars $c_{n,m}$. The multiplier a is independent of n and the coefficients are not functions of x .

Theorem 2. *Let $a \neq 0$ have degree d . An eventual scalar band recurrence (1) exists if and only if $aR \subset R$. When this holds, for every $n \in I$ with $n > D + d$ one has*

$$aq_n = \sum_{\substack{m \in I \\ n-d \leq m \leq n+d}} \frac{\langle aq_n, q_m \rangle}{\langle q_m, q_m \rangle} q_m. \quad (2)$$

Proof. First suppose (1) holds. For fixed $m \in I$, symmetry gives $\langle aq_m, q_n \rangle = \langle q_m, aq_n \rangle$. For sufficiently large n , the right side vanishes because every basis index in the recurrence is larger than m . Lemma 1 implies $aq_m \in R$, and therefore $aS \subset R$. Also, for $t \in T$, $\langle at, q_n \rangle = \langle t, aq_n \rangle = 0$

for all sufficiently large n , because the recurrence puts aq_n in S . The same lemma gives $aT \subset R$. Together these inclusions prove $aR \subset R$.

Conversely assume $aR \subset R$ and $n > D + d$. We first show that aq_n lies in S , not merely in R . For $t \in T$ write $at = s + t'$ with $s \in S$ and $t' \in T$. Then $\deg s \leq D + d < n$, and

$$\langle aq_n, t \rangle = \langle q_n, at \rangle = \langle q_n, s + t' \rangle = 0.$$

Consequently the T -component of $aq_n \in S \oplus T$ is zero. When $T = 0$ this conclusion is immediate. Expand aq_n in the algebraic basis of S . Its degree is $n + d$, so no term with $m > n + d$ occurs.

If $m < n - d$, write $aq_m = s_m + t_m$ in $S \oplus T$. The estimate

$$\deg s_m \leq \max\{m + d, D\} < n$$

gives $\langle aq_n, q_m \rangle = \langle q_n, s_m + t_m \rangle = 0$. This removes every term below $n - d$ and proves (2). $\square$

Corollary 3. *Some nonconstant real polynomial has an eventual scalar band recurrence if and only if R contains a nonzero polynomial ideal $M\mathbb{R}[x]$.*

Proof. If $R = V$, take $a = x$ and $M = 1$. Otherwise, when a nonconstant a preserves R , multiplication by a induces an endomorphism of the finite-dimensional space V/R . If χ is its characteristic polynomial, Cayley–Hamilton gives $\chi(a)V \subset R$. The polynomial $\chi(a)$ is nonzero because a is nonconstant. Conversely, if $MV \subset R$ with M nonconstant, then $MR \subset MV \subset R$, and Theorem 2 applies to $a = M$. A nonzero constant M forces $R = V$. $\square$

This is a structural criterion, not a decision algorithm for arbitrary functionals supplied by oracles. If a polynomial $M \neq 0$ with $MV \subset R$ is already known, the stabilizer

$$\text{Stab}(R) = \{a \in V : aR \subset R\}$$

is the inverse image of the algebra of residues in $V/(M)$ whose multiplication preserves the subspace $R/(M)$. Thus the remaining stabilizer calculation is finite-dimensional. The finite-quotient and Cayley–Hamilton argument used here is standard.

3 A constraint on a separated interval

Theorem 4. *Let*

$$\langle p, q \rangle = \int_0^1 p(x)q(x) dx, \quad \lambda(p) = \int_2^3 p(x) dx, \quad S = \ker \lambda.$$

Then S has a monic orthogonal basis $q_1, q_2, \dots$, with exactly the degree zero missing. For every nonconstant $a \in \mathbb{R}[x]$, there are infinitely many n for which $aq_n \notin S$. In particular, no such a has an eventual scalar band recurrence.

Proof. Since $\lambda(1) = 1$, the constrained space has codimension one and attains every positive degree. The basis therefore exists by the construction in Section 1.

Suppose, to the contrary, that $aq_n \in S$ for every $n \geq N$. The restriction of $p \mapsto \lambda(ap)$ to S is then represented by a finite orthogonal expansion: there is $r \in \text{span}\{q_1, \dots, q_{N-1}\}$ such that $\lambda(ap) = \langle p, r \rangle$ for all $p \in S$. Explicitly,

$$r = \sum_{1 \leq n < N} \frac{\lambda(aq_n)}{\langle q_n, q_n \rangle} q_n.$$

The difference $p \mapsto \lambda(ap) - \langle p, r \rangle$ vanishes on $\ker \lambda$, so it equals $c\lambda$ on V for some $c \in \mathbb{R}$. Consequently

$$\int_2^3 (a(x) - c)p(x) dx = \int_0^1 r(x)p(x) dx \quad (p \in V). \quad (3)$$

Take $p(x) = x^{2j}(a(x) - c)$. The left side is at least

$$4^j \int_2^3 (a(x) - c)^2 dx,$$

whereas the absolute value of the right side is bounded by $\int_0^1 |r(x)(a(x) - c)| dx$, independently of j . Thus $a = c$ on $[2, 3]$ and hence as a polynomial, a contradiction. The failure of eventual membership is failure at infinitely many indices. It even rules out an eventual expansion in S of unrestricted finite width at each index. $\square$

One can also apply Corollary 3 directly. Here $T = 0$: if $r \in T$, then $\langle p, r \rangle = c\lambda(p)$ for every polynomial p . Testing $p = x^{2j}$ bounds the left side independently of j , while $\lambda(x^{2j}) \geq 4^j$. Hence $c = 0$, and testing $p = r$ gives $r = 0$. Moreover, a nonzero ideal $MV \subset S$ would imply $\lambda(M^2) = 0$, contradicting positivity of its integral on $[2, 3]$.

Proposition 5. *The space S in Theorem 4 is proper and dense in $L^2([0, 1])$. No bounded Hilbert-space orthogonal projection has range exactly this algebraic space S .*

Proof. Put $b_j = \lambda(x^j)$ and $e_j = x^j/b_j$. Then $\lambda(e_j) = 1$, $b_j \geq 2^j$, and

$$\|e_j\|_2^2 = \frac{1}{(2j+1)b_j^2} \leq \frac{1}{(2j+1)4^j} \rightarrow 0.$$

For every polynomial p , the elements $p - \lambda(p)e_j$ belong to S and converge to p in $L^2([0, 1])$. Polynomial density proves density of S ; the constant 1 is not in S . The range of a bounded orthogonal projection is closed, proving the last assertion. $\square$

The projection onto the closure of this S is the identity. Proposition 5 identifies a limitation of interpreting the workshop's compression suggestion as a projection onto the algebraic space itself. It does not rule out finite-dimensional, oblique, or closed-subspace compression methods.

4 Why the polynomial complement matters

Keep the same inner product but impose $\lambda_0(p) = \int_0^1 p(x) dx = 0$. Then $T = \text{span}\{1\}$ and $R = V$. The constrained basis consists of the ordinary monic shifted Legendre polynomials q_n , $n \geq 1$. The usual three-term recurrence for multiplication by x holds for $n \geq 2$, but $xS \not\subset S$:

$$q_1 = x - \frac{1}{2}, \quad \lambda_0(xq_1) = \int_0^1 (x^2 - x/2) dx = \frac{1}{12}.$$

Thus $aS \subset S$ is too strong as a necessary condition for an eventual recurrence. For a recurrence valid at every basis index, invariance of S is necessary, and symmetry and degree counting make it sufficient. Passing to a tail replaces that space by $S + T$.

The accompanying exact-arithmetic program checks five constrained families through degree 18: the separated-interval example, the same-measure mean, the constraint $\int_0^1 x^2 p(x) dx = 0$, the condition $p'(0) = 0$, and $p(0) + p(1) = 0$. Two constructions agree on 90 basis polynomials:

constrained Gram–Schmidt and a closed shifted-Legendre formula with a rank-one kernel correction. There are 972 constraint and orthogonality checks, 628 recurrence-coefficient checks in the four positive controls, and 18 nonzero x -defects in the negative example. Additional tests cover the degree gaps $\{1, 4, 9\}$, the density construction, and four deliberate corruptions. These finite checks are regression tests, not proofs of the infinite statements above.

The generic recurrence obstruction and the stabilizer method are prior ideas, as credited in Section 1. The formulation emphasized here is the exact criterion using $S + S^\perp$ for a tail recurrence and the explicit separated-interval obstruction for every nonconstant fixed multiplier. No absolute novelty is asserted. AI assistance was used in the mathematical derivation, literature search, exact-arithmetic checks, and preparation of this manuscript. The arguments have been self-audited; this is not a claim of independent peer review or formal proof certification.

References

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