

A Gaussian Obstruction to Uniform Mesh Bounds for Differential Operators

Alper Ferudun*

10 October 2026

Abstract

We classify the nonzero finite-order differential operators with real polynomial coefficients that preserve hyperbolicity and retain a positive uniform fraction of additive zero mesh on the whole Laguerre–Pólya class: they are exactly the nonzero scalar multiples of the identity. For every other such operator and every prescribed positive input mesh, the infimum of the output mesh is zero. The inputs may have exactly three simple zeros. The proof uses a strengthening Gaussian factor and freezes the leading coefficient at a nonsingular center. For an arbitrary fixed operator of positive order, without a preservation assumption, we also compute the common drift and next common dilation of the resulting local Hermite zero cluster. The basic Gaussian obstruction is an antecedent, not a novelty claim. This finite-operator result does not settle all zero-spacing questions for entire functions. The manuscript is unrefereed.

1 Uniform mesh retention and its domain

Let $\mathcal{LP}$ be the class of nonzero entire functions that are locally uniform limits of real polynomials with only real zeros. For $f \in \mathcal{LP}$ with at least two zeros, let $\text{mesh}(f)$ be the infimum of the distances between consecutive real zeros, counted with multiplicity. In particular, a multiple zero gives mesh zero. We use the statistic only when the indicated functions have at least two zeros.

Write $D = d/dx$. A nonzero finite differential operator in normal order is

$$T = \sum_{j=0}^r p_j(x) D^j, \quad p_j \in \mathbb{R}[x], \quad p_r \neq 0. \quad (1)$$

The derivatives act on the input, not on the coefficients to their left. We call T a *hyperbolicity preserver* if it sends every real-rooted real polynomial to a real-rooted polynomial or zero. For such a T and $b > 0$, define

$$I_T(b) = \inf\{\text{mesh}(Tf) : f \in \mathcal{LP}, \text{mesh}(f) = b, Tf \neq 0, Tf \text{ has at least two zeros}\}. \quad (2)$$

The proof below shows that this set is nonempty for every $b > 0$.

Theorem 1.1. *Let T be a nonzero operator of the form (1) preserving hyperbolicity. The following are equivalent:*

1. *T is multiplication by a nonzero real constant.*

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

2. There exists $C_T > 0$ such that $\text{mesh}(Tf) \geq C_T \text{mesh}(f)$ for every $f \in \mathcal{LP}$ with finite positive mesh and nonzero output having at least two zeros.
3. $I_T(b) > 0$ for some $b > 0$.

For scalar T , $I_T(b) = b$ for every $b > 0$. For every nonscalar T , $I_T(b) = 0$ for every $b > 0$, even if the infimum is restricted to inputs with exactly three simple zeros.

The input domain is essential. For real-rooted polynomials, the classical mesh theorem of Riesz and its constant-coefficient extension give positive results; see Katkova–Shapiro–Vishnyakova [4, Theorem 2]. Farmer–Rhoades [2, Theorems 2.1.1 and 2.3.1] treat related polynomial and order-one entire-function settings. The witnesses used here have a quadratic Gaussian factor. Closures of fixed-mesh polynomial classes, studied by Golitsyna [3], must not be identified with all of $\mathcal{LP}$ merely because their limiting functions have separated zeros.

Gaussian failures for differentiation were known earlier, for example in Vishnyakova’s 2015 lecture slides [6]. The frozen UnsolvedMath research record [5] already gives the cubic-Gaussian obstruction for D^r , for every $r \geq 1$. We credit that construction. Our purpose is to extract the coefficient-general classification and the explicit common correction coefficients of Section 2. The Hermite scaling and implicit-function method are classical; no absolute-priority claim is made.

The AIM source recorded in [5], Problem 5.2 of the 2011 workshop on stability, hyperbolicity and zero localization, asks broadly about higher-order differential operators and zero spacing in $\mathcal{LP}$. Theorem 1.1 addresses the precise finite-operator, additive-mesh benchmark above, not that entire research question.

2 A local Hermite cluster and its corrections

Fix an operator (1) of order $r \geq 1$, a real c with $A = p_r(c) \neq 0$, and $b > 0$. Set

$$f_a(x) = (x - c)((x - c)^2 - b^2)e^{-a(x-c)^2}, \quad a > 0. \quad (3)$$

For every a , this function has exactly the three simple zeros $c - b, c, c + b$ and mesh b . It belongs to $\mathcal{LP}$: the real-rooted polynomials

$$(x - c)((x - c)^2 - b^2) \left(1 - \frac{a(x - c)^2}{N}\right)^N$$

converge locally uniformly to it. Their repeated zeros are allowed; their meshes are not asserted to stay positive.

Let H_n be the physicists’ Hermite polynomial,

$$H_n(y) = (-1)^n e^{y^2} D_y^n e^{-y^2},$$

and write its zeros as $\eta_1 < \cdots < \eta_n$.

Theorem 2.1. Put $n = r + 1$, $q = p_{r-1}/p_r$ near c , and

$$C = \begin{cases} p_{r-2}(c), & r \geq 2, \\ 0, & r = 1. \end{cases}$$

Define

$$\alpha = \frac{q(c)}{2n}, \quad \kappa = -\frac{1}{2b^2} + \frac{q'(c)}{2n} - \frac{C}{2n(n-1)A} + \frac{q(c)^2}{4n^2}. \quad (4)$$

For all sufficiently large a , Tf_a has n distinct simple real zeros near c with expansions

$$x_j(a) = c + \frac{\eta_j}{\sqrt{a}} + \frac{\alpha}{a} + \frac{\kappa\eta_j}{a^{3/2}} + O(a^{-2}). \quad (5)$$

Consequently, for $1 \leq j < n$,

$$x_{j+1}(a) - x_j(a) = \frac{\eta_{j+1} - \eta_j}{\sqrt{a}} + \frac{\kappa(\eta_{j+1} - \eta_j)}{a^{3/2}} + O(a^{-2}). \quad (6)$$

No hyperbolicity-preserver assumption is needed. Other zeros need not be real. The threshold and error constants depend on the fixed operator, c and b .

Proof. Put $\epsilon = a^{-1/2}$, $x = c + \epsilon y$, and define polynomial operators

$$E_j(P) = e^{y^2} D_y^j (P e^{-y^2}), \quad E_0(P) = P, \quad E_{j+1}(P) = E_j(P)' - 2y E_j(P).$$

The exact identity

$$\begin{aligned} G(\epsilon, y) &:= e^{y^2} \epsilon^{r-1} (Tf_{\epsilon^{-2}})(c + \epsilon y) \\ &= \sum_{j=0}^r \epsilon^{r-j} p_j(c + \epsilon y) E_j(-b^2 y + \epsilon^2 y^3) \end{aligned} \quad (7)$$

extends polynomially to $\epsilon = 0$. Rodrigues' formula gives

$$E_j(y) = \frac{(-1)^j}{2} H_{j+1}(y), \quad E_j(y^3) = (-1)^j \left(\frac{H_{j+3}(y)}{8} + \frac{3H_{j+1}(y)}{4} \right). \quad (8)$$

For the second identity, use $y^3 = H_3(y)/8 + 3H_1(y)/4$ and $E_j(H_m) = (-1)^j H_{m+j}$.

Write $B = p_{r-1}(c)$ and $K = (-1)^{r+1} b^2 A/2$. Expanding (7) to second order gives

$$G/K = H_n + \epsilon U + \epsilon^2 V + O(\epsilon^3), \quad (9)$$

uniformly for y in any fixed compact set, where

$$U(y) = \frac{p'_r(c)}{A} y H_n(y) - \frac{B}{A} H_{n-1}(y), \quad (10)$$

$$\begin{aligned} V(y) &= \frac{p''_r(c)}{2A} y^2 H_n(y) - \frac{p'_{r-1}(c)}{A} y H_{n-1}(y) + \frac{C}{A} H_{n-2}(y) \\ &\quad - \frac{1}{b^2} \left(\frac{H_{n+2}(y)}{4} + \frac{3H_n(y)}{2} \right). \end{aligned} \quad (11)$$

The C term is zero if $r = 1$.

For completeness, the Hermite zeros are real and simple. Rodrigues' formula and integration by parts show that H_n is orthogonal to every polynomial of degree less than n for the positive weight e^{-y^2} on $\mathbb{R}$. If it had fewer than n real sign-changing zeros, multiply it by the product of the distinct linear factors at those zeros. The product has constant sign on $\mathbb{R}$, is not zero, and has nonzero weighted integral, contradicting orthogonality. Since $\deg H_n = n$, all its zeros are real and simple.

The analytic implicit-function theorem applied to $G(0, \eta_j) = 0$ therefore gives a simple zero branch $y_j(\epsilon)$ with $y_j(0) = \eta_j$. Real coefficients and uniqueness make the branch real for real ϵ . Distinct neighborhoods and nonzero derivatives persist for sufficiently small ϵ . There are finitely many branches, so a common threshold can be chosen.

At a zero η of H_n , the Hermite recurrence and differential equation give

$$\begin{aligned} H'_n(\eta) &= 2nH_{n-1}(\eta), & H''_n(\eta) &= 2\eta H'_n(\eta), \\ H_{n-2}(\eta) &= \frac{\eta}{n-1}H_{n-1}(\eta), & H_{n+2}(\eta) &= -4n\eta H_{n-1}(\eta). \end{aligned} \quad (12)$$

Write $y_j = \eta + \alpha_j\epsilon + \beta_j\epsilon^2 + O(\epsilon^3)$. Substitution in (9) gives

$$\alpha_j = -\frac{U(\eta)}{H'_n(\eta)} = \frac{B}{2nA}, \quad (13)$$

$$\begin{aligned} \beta_j &= -\frac{V(\eta) + \alpha_j U'(\eta) + \alpha_j^2 H''_n(\eta)/2}{H'_n(\eta)} \\ &= \eta \left[-\frac{1}{2b^2} + \frac{p'_{r-1}(c)}{2nA} - \frac{Bp'_r(c)}{2nA^2} - \frac{C}{2n(n-1)A} + \frac{B^2}{4n^2A^2} \right] = \kappa\eta. \end{aligned} \quad (14)$$

Thus the first scaled correction is independent of j , and the next is proportional to η_j . Multiplying by ϵ proves (5); subtraction proves (6). $\square$

3 Proof of the mesh classification

Proof of Theorem 1.1. Every finite differential operator with polynomial coefficients is continuous for locally uniform convergence of entire functions, by Cauchy's formula for its finitely many derivatives. Thus a polynomial hyperbolicity preserver maps $\mathcal{LP}$ to $\mathcal{LP}$ or zero.

Suppose first that $r \geq 1$. Choose $c \in \mathbb{R}$ outside the finite zero set of p_r and use (3). For all sufficiently large a , Theorem 2.1 gives $r+1 \geq 2$ simple real output zeros, so Tf_a is nonzero. All its zeros are real because $Tf_a \in \mathcal{LP}$. A distance between two selected zeros bounds the full mesh from above: if they are not consecutive, one of the intervening gaps is even smaller. Consequently,

$$0 \leq \text{mesh}(Tf_a) \leq \frac{\eta_{j+1} - \eta_j}{\sqrt{a}} + O(a^{-3/2}) \longrightarrow 0, \quad \text{mesh}(f_a) = b.$$

This proves $I_T(b) = 0$ for every $b > 0$, using three-simple-zero inputs.

If $r = 0$, then T is multiplication by p_0 . Applying T to 1 shows that p_0 is real-rooted. If it is nonconstant, choose a root t and, for $0 < h < b$, put

$$g_h(x) = \prod_{k=-1}^1 (x - t - h - kb).$$

This polynomial has exactly three simple zeros and mesh b , whereas $p_0 g_h$ has zeros t and $t+h$. Its mesh is at most h , possibly already zero because of multiplicities. Letting $h \downarrow 0$ proves $I_T(b) = 0$ also in this case. For a nonzero constant p_0 , multiplication preserves every zero and multiplicity, giving $I_T(b) = b$ and $C_T = 1$. The equivalences follow, and the explicit witnesses prove nonemptiness of all infima used in (2). $\square$

4 Examples and boundaries

For $T = D^r$, the coefficients in (4) are $\alpha = 0$ and $\kappa = -1/(2b^2)$. The leading mesh obstruction is the inherited example of [5]; the formulas record its next local correction. For $T = D + \lambda$, $\lambda \in \mathbb{R}$, one has $n = 2$,

$$\alpha = \lambda/4, \quad \kappa = -1/(2b^2) + \lambda^2/16.$$

The displacement of order a^{-1} cancels from the gap in both cases. The local statement also applies to $(x^2 + 1)D^r$, which is not a hyperbolicity preserver: its complex factor shows why a real cluster alone says nothing about global real-rootedness.

Higher multiplicity at the input center cannot be handled by merely shifting a single Hermite index. For example, $E_1(y^2) = 2y - 2y^3$ is not proportional to $H_3(y) = 8y^3 - 12y$. The simple central zero in (3) avoids this issue.

Translations preserve mesh but are not finite-order differential operators. Allowing arbitrary entire coefficients also changes the classification: multiplication by e^{-x^2} preserves $\mathcal{LP}$ and all zeros without being a scalar. The zero operator is excluded, and no uniform expansion as r grows or $p_r(c)$ approaches zero is asserted. Neither logarithmic mesh nor all Gaussian-free domains are classified here. These distinctions also separate the theorem from mesh questions for finite-difference operators [1].

5 Exact regression checks

The accompanying standard-library Python checker uses rational arithmetic. It checks (7) independently by differentiating before rescaling, and checks (10)–(14) by polynomial identities and reduction modulo H_n . Its 450 cases use $r = 1, \dots, 10$, $c \in \{-1, 1, 2\}$, $b \in \{1/2, 1, 2\}$, and the five families D^r , $(D + 1)^r$, $D^r \circ M_{x+3}$, $(xD + 1)^r$, and $(x^2 + 1)D^r$, where M_{x+3} denotes multiplication by $x + 3$. The last family deliberately tests the local theorem without preservation.

For each case, rational isolating intervals for the Hermite zeros and explicit endpoint perturbation bounds certify a real root in each interval for an entire stated range $0 < \epsilon \leq \epsilon_{\max}$. Normal and optimized runs give identical outputs with 9,505 checks and six negative controls. These finite checks support the formulas; the proofs above, not the enumeration, establish the general statements.

The author used AI tools to develop and check the arguments, compare literature, and prepare the manuscript and reproducibility code. The claims have been self-audited within their stated scope; they have not undergone independent peer review or proof-assistant verification.

References

- [1] P. Brändén, I. Krasikov and B. Shapiro, Elements of Pólya–Schur theory in the finite difference setting, *Proc. Amer. Math. Soc.* **144** (2016), 4831–4843. [doi:10.1090/proc/13115](https://doi.org/10.1090/proc/13115).
- [2] D. W. Farmer and R. C. Rhoades, Differentiation evens out zero spacings, *Trans. Amer. Math. Soc.* **357** (2005), 3789–3811. [doi:10.1090/S0002-9947-05-03721-9](https://doi.org/10.1090/S0002-9947-05-03721-9).
- [3] M. Golitsyna, Analytic closure of sets of real hyperbolic polynomials with separated roots, *Eur. J. Math.* **1** (2015), 641–653. [doi:10.1007/s40879-015-0047-3](https://doi.org/10.1007/s40879-015-0047-3).
- [4] O. Katkova, B. Shapiro and A. Vishnyakova, Multiplier sequences and logarithmic mesh, *C. R. Math. Acad. Sci. Paris* **349** (2011), 35–38. [doi:10.1016/j.crma.2010.11.031](https://doi.org/10.1016/j.crma.2010.11.031).
- [5] ulamai, *UnsolvedMath*, version 1.6.0, record AIM-ANALYSIS-0146 and accompanying machine-generated research report in `research_results.json`, unrefereed. Frozen commit `c6e7f41b4eb6ce876feb83e7eb626be1b243fd7e`. [Frozen dataset snapshot](https://zenodo.org/record/13115).
- [6] A. Vishnyakova, *Zero separation of polynomials and hyperbolicity preserving linear operators*, lecture slides, Shanghai, 2015. <https://www.researchgate.net/publication/283146858>.