# Sharp Zero Regions and Hurwitz Criteria for BMV Trace Polynomials

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**Abstract**

For positive-definite Hermitian matrices $A, B$, we study the zeros of $P_m(z) = \mathrm{Tr}((A + zB)^m)$. Loewner bounds $lB \le A \le uB$ give a sharp dimension-independent zero region: the union of two explicit closed disks. Every point of this region is attained by a commuting two-dimensional pair, while a nonreal boundary zero forces an endpoint block decomposition and commutativity. Optimizing the associated angle gives the optimal universal Hurwitz guarantee $u/l < \tan^2(\pi/4 + \pi/(2m))$ for $m \ge 3$. For arbitrary two-dimensional pairs, including noncommuting ones, a quadratic factorization yields simplicity, an exact stability criterion in terms of $\mathrm{Tr}(AB)/(\mathrm{Tr}\, A\, \mathrm{Tr}\, B)$, and a count of right-half-plane zeros. We also classify real zeros in every dimension. These are explicit zero-location results motivated by an open-ended AIM problem, not a new proof of the BMV coefficient theorem or a classification for every fixed higher-dimensional pair.

## 1 Setting and scope

Let $A, B$ be positive-definite Hermitian $n \times n$ matrices and let $m \ge 1$ be an integer. Define

$$P_m(z) = \mathrm{Tr}\big((A + zB)^m\big). \tag{1}$$

Matrix powers are taken before the trace. This is a real polynomial of degree $m$, with leading coefficient $\mathrm{Tr}(B^m) > 0$: it is real-valued for real $z$. We call it strictly Hurwitz stable when all its zeros have strictly negative real part.

Fix $0 < l \le u$ with

$$lB \le A \le uB. \tag{2}$$

The sharp endpoints are the least and greatest eigenvalues of $B^{-1/2}AB^{-1/2}$. This congruence identifies the bounds; it does *not* turn (1) into the trace of a power of $B^{-1/2}AB^{-1/2} + zI$.

The AIM problem list for the 2011 workshop on stability, hyperbolicity, and zero localization asks, in Problem 2.1 attributed to O. Holtz, to study the zeros of these polynomials [1, 2]. The question is open-ended. We prove a sharp universal locus under prescribed bounds, its boundary rigidity, an optimal uniform stability threshold, and a complete factorization and half-plane count in dimension two. The universal locus is a union of zero sets as the pair varies, not a formula for the roots of a fixed pair in every dimension.

Coefficient nonnegativity is a different question. The polynomial formulation is related to the original Bessis–Moussa–Villani conjecture by Lieb–Seiringer [3]; that conjecture was proved by Stahl [4], with a subsequent treatment by Eremenko [5]. We do not re-prove it. In particular, nonnegative coefficients do not imply Hurwitz stability. General polynomial root-localization and Hurwitz techniques are discussed in [6].

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Our proofs use standard Schur triangularization, cancellation in a half-plane, and classical two-variable power-sum identities. The latter are instances of the Girard–Waring/Chebyshev framework [7]. The frozen UnsolvedMath research record [8] already contains the commuting $2 \times 2$ Möbius family used below; we recover and credit that starting case rather than claim its discovery. The noncommuting factorization and the all-dimensional equality argument are given explicitly. A bounded primary-source search did not identify a matching formulation of the combined results; folklore and unlocated prior art are not excluded.

If $l = u$, then $A = lB$ and

$$P_m(z) = \operatorname{Tr}(B^m)(z + l)^m. \tag{3}$$

For $m = 1$ there is just the negative zero $-\operatorname{Tr} A/\operatorname{Tr} B$. Unless stated otherwise, the localization results below assume $l < u$ and $m \geq 2$.

## 2 A sharp universal zero region

For $y > 0$ and $z = x + iy$, put

$$\alpha = \arg(z + u), \qquad \beta = \arg(z + l), \qquad \delta(z) = \beta - \alpha \in (0, \pi), \tag{4}$$

where both arguments belong to $(0, \pi)$. In the lower half-plane use $\delta(z) = \delta(\overline{z})$. Define

$$\mathcal{R}_{l,u,m} = \left\{ x + iy : (x + l)(x + u) + y^2 \leq (u - l)|y| \cot \frac{\pi}{m} \right\}. \tag{5}$$

Completing the square shows that this set is the union of the closed disks with centers and common radius

$$-\frac{l + u}{2} \pm i\frac{u - l}{2} \cot \frac{\pi}{m}, \qquad \frac{u - l}{2} \csc \frac{\pi}{m}. \tag{6}$$

The disks coincide when $m = 2$. Their union, not their intersection, is intended. Its intersection with the real axis is $[-u, -l]$.

**Theorem 1** (Localization and universal attainability)**.** *Every zero of $P_m$ belongs to $\mathcal{R}_{l,u,m}$. Equivalently, each nonreal zero satisfies $\delta(z) \geq \pi/m$, and each real zero lies in $[-u, -l]$. Conversely, every point of $\mathcal{R}_{l,u,m}$ is a zero for some commuting positive-definite $2 \times 2$ pair satisfying (2).*

*Proof.* First take $y > 0$. Choose a unitary Schur basis $q_1, \ldots, q_n$ for $H = A + zB$. The diagonal eigenvalues of its triangular Schur form are

$$\lambda_j = q_j^* H q_j = b_j(c_j + z), \qquad b_j = q_j^* B q_j > 0, \qquad c_j = \frac{q_j^* A q_j}{q_j^* B q_j} \in [l, u].$$

Their arguments lie in $[\alpha, \beta]$, and none is zero. If $m\delta < \pi$, rotation by $e^{-im(\alpha+\beta)/2}$ makes every $\lambda_j^m$ have strictly positive real part. Thus $P_m(z) = \sum_j \lambda_j^m \neq 0$. The trace identity holds for upper triangular matrices and does not require diagonalizability of $H$.

The dot product of $z + l$ and $z + u$ is $D = (x + l)(x + u) + y^2$, and their cross-product magnitude is $(u - l)y$. Consequently

$$\cot \delta = \frac{D}{(u - l)y}.$$

Cotangent is decreasing on $(0, \pi)$, which proves the equivalence with (5) without an arctangent branch restriction. Conjugation handles $y < 0$. For real $x > -l$ the matrix $A + xB$ is positive definite, while for $x < -u$ it is negative definite. Its trace power is nonzero in both cases.

For attainability take a nonreal point in the region, first with $y > 0$. The continuous strictly decreasing function $c \mapsto \arg(z+c)$ on $[l,u]$ has total variation at least $\pi/m$. Choose $c, d \in [l,u]$ so that $\arg(z+c) - \arg(z+d) = \pi/m$, and set

$$B = \operatorname{diag}\left(\frac{1}{|z+c|}, \frac{1}{|z+d|}\right), \qquad A = \operatorname{diag}\left(\frac{c}{|z+c|}, \frac{d}{|z+d|}\right).$$

Both matrices are positive definite and satisfy the required bounds. The eigenvalues of $A + zB$ have modulus one and their $m$th powers are opposite. Thus $P_m(z) = 0$. Conjugation gives the lower-half-plane construction. A real $x \in [-u, -l]$ is obtained by taking $A = (-x)B$. $\square$

The attaining matrices may depend on the chosen point. Moreover, the universal-locus statement prescribes bounds $l, u$, not that both must be the actual extreme generalized eigenvalues of every witness.

**Theorem 2** (Boundary rigidity)**.** *Suppose $z$ is nonreal, $\delta(z) = \pi/m$, and $P_m(z) = 0$. Then there is an orthogonal decomposition into two nonzero summands such that*

$$A = lB_l \oplus uB_u, \qquad B = B_l \oplus B_u, \tag{7}$$

*where $B_l, B_u$ are positive definite, and*

$$|z+l|^m \operatorname{Tr}(B_l^m) = |z+u|^m \operatorname{Tr}(B_u^m). \tag{8}$$

*Conversely, (7) and (8) give a zero at such a boundary point. In particular, a noncommuting pair has all its nonreal zeros in the strict interior of (5).*

*Proof.* It suffices to consider $y > 0$. Use the Schur basis in the preceding proof. At $m\delta = \pi$, the rotated powers have nonnegative real parts. Since they sum to zero, every real part vanishes. Thus every $c_j$ equals $l$ or $u$: interior angles have strictly positive cosine after the rotation.

If $c_j = l$, then $q_j^*(A - lB)q_j = 0$. Positivity of $A - lB$ implies $(A - lB)q_j = 0$. Similarly $c_j = u$ gives $(A - uB)q_j = 0$. In this orthonormal basis write $A' = B'D$, where $D$ is diagonal with entries in $\{l, u\}$. Hermiticity gives $B'D = DB'$. Hence $B'$ is block diagonal for the two values of $D$, and $A'$ is proportional to it on each block. Both blocks occur, since a single ray cannot give a zero sum of nonzero powers. This proves (7); it also proves ordinary commutativity, not merely simultaneous congruence.

The trace polynomial is now

$$(z+l)^m \operatorname{Tr}(B_l^m) + (z+u)^m \operatorname{Tr}(B_u^m).$$

The two complex factors have phase difference $\pi$. Cancellation is therefore equivalent to (8), proving the converse too. $\square$

## 3 An optimal uniform Hurwitz threshold

**Theorem 3.** *For $m \geq 3$, set*

$$K_m = \tan^2\left(\frac{\pi}{4} + \frac{\pi}{2m}\right). \tag{9}$$

*If $u/l < K_m$, then $P_m$ is strictly Hurwitz stable. If $u/l = K_m$, all zeros are in the closed left half-plane. In this equality case, imaginary-axis zeros can only be $z = \pm i\sqrt{lu}$; they occur precisely when (7) holds and*

$$\frac{\operatorname{Tr}(B_l^m)}{\operatorname{Tr}(B_u^m)} = \left(\frac{u}{l}\right)^{m/2}. \tag{10}$$

*In particular, noncommuting pairs remain strictly stable at the threshold. For every prescribed $0 < l < u$ with $u/l > K_m$, there is a commuting $2 \times 2$ example with actual extreme generalized eigenvalues $l, u$ and a right-half-plane zero. For $m = 2$ every positive-definite pair is strictly Hurwitz stable.*

*Proof.* For $x \geq 0$ and $y > 0$, the angle in (4) is acute and

$$\tan\delta = \frac{(u-l)y}{(x+l)(x+u)+y^2}.$$

For each fixed $x$ its unique maximum is at $y = \sqrt{(x+l)(x+u)}$. The resulting maximum decreases strictly as $x$ increases. Thus the unique maximizing point in the upper closed right half-plane is $i\sqrt{lu}$, and

$$\delta_{\max} = 2\arctan\frac{\sqrt{u}-\sqrt{l}}{\sqrt{u}+\sqrt{l}}. \tag{11}$$

The inequality $\delta_{\max} < \pi/m$ is equivalent to $u/l < K_m$. Theorem 1 proves stability. At equality, only the two conjugate maximizing points can be zeros in the closed right half-plane. There $|z+u|/|z+l| = \sqrt{u/l}$, so Theorem 2 gives the stated characterization. When $m = 2$, the angle is always less than $\pi/2$ in that half-plane, which proves strict stability directly.

For sharpness put $r = \sqrt{u/l} > 1$, $\mu = \sqrt{lu}$, and take

$$A = \mu\,\mathrm{diag}(1, r), \qquad B = \mathrm{diag}(r, 1). \tag{12}$$

The generalized eigenvalues are $\mu/r = l$ and $\mu r = u$. Its zeros are $z = \mu w$, where

$$w = \frac{r\zeta - 1}{r - \zeta}, \qquad \zeta^m = -1. \tag{13}$$

Indeed, the trace polynomial divided by $\mu^m$ is $(1+rw)^m + (r+w)^m$. For $\zeta = e^{i\theta}$,

$$\Re w = \frac{(r^2+1)\cos\theta - 2r}{|r - e^{i\theta}|^2}.$$

Taking $\theta = \pi/m$, this is positive exactly when $r > \tan(\pi/4 + \pi/(2m))$. This proves sharpness with the actual prescribed extrema, not just with weaker bounds. $\square$

# 4 Complete factorization in dimension two

Assume now $n = 2$, with no commutativity hypothesis. Set

$$\begin{aligned} a &= \mathrm{Tr}\,A, & b &= \mathrm{Tr}\,B, & d_0 &= \det A, & d_2 &= \det B, \\ d_1 &= ab - \mathrm{Tr}(AB), & s(z) &= a + bz, & d(z) &= d_0 + d_1 z + d_2 z^2. \end{aligned} \tag{14}$$

Thus $s(z) = \mathrm{Tr}(A + zB)$ and $d(z) = \det(A + zB)$. In particular $d_1 = \mathrm{Tr}(\mathrm{adj}(A)B) > 0$.

**Theorem 4** (Quadratic factors and simplicity)**.** *Put $r = \lfloor m/2 \rfloor$, $\epsilon = m - 2r$, and*

$$\theta_j = \frac{(2j+1)\pi}{m}, \qquad \gamma_j = 2(1+\cos\theta_j), \qquad 0 \leq j < r.$$

*Then*

$$P_m(z) = s(z)^\epsilon \prod_{j=0}^{r-1}\left(s(z)^2 - \gamma_j d(z)\right). \tag{15}$$

*If $A$ is not proportional to $B$, each quadratic factor has two simple nonreal conjugate zeros; these factors are pairwise coprime. For odd $m$ there is additionally the simple zero $-a/b$, shared with none of the quadratics. Thus all $m$ zeros are simple. If $A = cB$, there is instead just the zero $-c$ with multiplicity $m$.*

*Proof.* For two complex numbers $\lambda_1, \lambda_2$, factor $\lambda_1^m + \lambda_2^m$ using the $m$ roots of $\zeta^m = -1$. Pairing conjugate roots, and retaining the root $-1$ when $m$ is odd, gives

$$(\lambda_1 + \lambda_2)^\epsilon \prod_{j=0}^{r-1} (\lambda_1^2 - 2\cos\theta_j \lambda_1\lambda_2 + \lambda_2^2).$$

Each bracket is $(\lambda_1 + \lambda_2)^2 - \gamma_j \lambda_1 \lambda_2$. Substitution of the trace and determinant proves (15), including repeated eigenvalues, by a symmetric-polynomial identity.

The $j$th factor is

$$(b^2 - \gamma_j d_2)z^2 + (2ab - \gamma_j d_1)z + (a^2 - \gamma_j d_0). \tag{16}$$

Since $0 < \gamma_j < 4$, $a^2 \geq 4d_0$, and $b^2 \geq 4d_2$, its constant and leading coefficients are strictly positive. For real $z$, the matrix $A + zB$ has real eigenvalues. Because $|\cos\theta_j| < 1$, the corresponding quadratic form $\lambda_1^2 - 2\cos\theta_j \lambda_1 \lambda_2 + \lambda_2^2$ is positive unless both eigenvalues vanish. A Hermitian matrix with both eigenvalues zero is zero, which would force proportionality of $A, B$. Thus, for a nonproportional pair, the quadratic has no real zero and has strictly negative discriminant.

If two different factors share a zero, subtracting their equations gives $d(z) = 0$, hence $s(z) = 0$. But $s(z) = 0$ makes $z = -a/b$ real. The resulting Hermitian matrix has trace and determinant zero and must be zero, again contradicting nonproportionality. The same reasoning excludes a common zero with the odd linear factor. The proportional case follows directly from (3). □

**Theorem 5** (Exact stability criterion and half-plane count)**.** *For $m \geq 2$ and $n = 2$, put*

$$q = \frac{\operatorname{Tr}(AB)}{\operatorname{Tr} A \operatorname{Tr} B} \in (0, 1).$$

*Then $P_m$ is strictly Hurwitz stable if and only if*

$$q > \frac{\cos(\pi/m)}{1 + \cos(\pi/m)}. \tag{17}$$

*For a nonproportional pair its number of zeros in the open right half-plane, counted with multiplicity, is exactly*

$$2\,\#\left\{0 \leq j < \lfloor m/2 \rfloor : \cos\theta_j > \frac{q}{1-q}\right\}. \tag{18}$$

*An equality for an index $j$ gives a purely imaginary pair; indices with the reversed strict inequality give left-half-plane pairs. The possible odd real zero is negative. At equality in (17) there is one imaginary pair and all other zeros are in the open left half-plane. The maximum possible number of right-half-plane zeros among positive-definite $2 \times 2$ pairs is*

$$2\left\lfloor \frac{m+1}{4} \right\rfloor. \tag{19}$$

*Proof.* Positivity of $\operatorname{Tr}(AB)$ and $d_1 = ab - \operatorname{Tr}(AB)$ gives $0 < q < 1$. For a real quadratic $Lz^2 + Mz + C$ with $L, C > 0$ and negative discriminant, both roots have real part $-M/(2L)$. In (16), the middle coefficient is

$$2ab((1 + \cos\theta_j)q - \cos\theta_j).$$

Its sign gives the complete count and the equality cases. Since $\cos\theta_0 = \cos(\pi/m)$ is the largest cosine, all the middle coefficients are positive exactly under (17). For proportional pairs, $q = \operatorname{Tr}(B^2)/(\operatorname{Tr} B)^2 \geq 1/2$, whereas the right side of (17) is less than $1/2$, so the same criterion remains valid. If equality holds for a later index rather than $j = 0$, the preceding indices contribute right-half-plane roots; they are not assumed stable.

Only indices with $\theta_j < \pi/2$ can contribute to (18), since $q/(1-q) > 0$. Their number is $\lfloor (m+1)/4 \rfloor$. For $A = \operatorname{diag}(1, r)$ and $B = \operatorname{diag}(r, 1)$, $q = 2r/(1+r)^2 \longrightarrow 0$ as $r \longrightarrow \infty$. Every such index therefore contributes for sufficiently large $r$, which proves (19). □

The trace-ratio criterion is restricted to dimension two. For example, $A = B = I_3$ and $m = 3$ give $q = 1/3$, equality in the displayed threshold, but $P_3(z) = 3(1+z)^3$ is strictly stable. Thus a verbatim extension of (17) to higher dimensions would be false.

## 5 Real zeros in arbitrary dimension

**Theorem 6.** *For even $m$, $P_m$ has a real zero if and only if $A = cB$ for some $c > 0$; then $-c$ is its only zero and has multiplicity $m$. For odd $m$, $P_m$ has exactly one real zero. It is simple unless $A = cB$ and $m > 1$, in which case its multiplicity is $m$. For nonproportional pairs the odd real zero lies strictly between $-u$ and $-l$ when $l, u$ are the actual extreme generalized eigenvalues.*

*Proof.* For real $x$, $H = A + xB$ is Hermitian. An even power has nonnegative trace, with equality precisely when $H = 0$. This proves the even claim. For odd $m$, cyclic invariance of trace gives

$$P'_m(x) = m \operatorname{Tr}\bigl(B(A + xB)^{m-1}\bigr).$$

For $m > 1$ this is strictly positive when $H \neq 0$, because a nonzero even power is positive semidefinite and $B$ is positive definite. For $m = 1$ it is $\operatorname{Tr} B > 0$. In the nonproportional case $H$ never vanishes, so strict increase, together with the signs at infinity, gives exactly one simple root. At the sharp endpoints, $A - uB$ and $A - lB$ are nonzero negative and positive semidefinite matrices respectively, giving strict signs for odd powers. The proportional case is explicit. □

## 6 Reproducibility and limitations

The supplied Python checker compares the factorization with direct noncommutative expansion of matrix polynomials in exact cyclotomic quotient rings. Its 1580 finite diagnostics consist of 486 factorization comparisons, 450 rational squarefreeness checks, 640 odd-derivative and equality checks, and four further boundary or invalid-generalization examples. Isolated Python 3.9 and 3.13 runs produce identical reports. These finite tests are not a proof of the universal statements.

A separate seeded floating-point experiment checks 3168 roots in dimensions two through seven, 544 dimension-two root counts, 1015 constructed witnesses, 23 boundary witnesses, and 84 threshold cases. It is not interval-certified and is not used in the proofs. The verification report also records a corrected checker exception and a pre-publication clarification of the imaginary-axis count.

The results require strictly positive-definite matrices; no singular positive-semidefinite extension is asserted. The universal locus ranges over pairs with given Loewner bounds, and does not classify the roots of every fixed higher-dimensional noncommuting pair. The source's open-ended programme therefore remains broader than these theorems. No absolute priority claim is made.

This unrefereed note was prepared with AI assistance in derivation, literature search, exact-arithmetic checking, and exposition. The supporting record contains an originating-researcher self-audit; independent peer review and proof-assistant verification are not claimed.

## References


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