

# Coordinate Subcubes of Weaver Partitions at the MSS Bound

Alper Ferudun\*

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## Abstract

Let  $v_1, \dots, v_m \in \mathbb{C}^d$  be a Parseval frame with squared norms at most  $\alpha$ . At the historical Marcus–Spielman–Srivastava discrepancy bound  $\sqrt{8\alpha} + 2\alpha$ , we exhibit an entire coordinate subcube of good signings. If  $0 < \alpha < (3 - 2\sqrt{2})/2$  and  $r = \lceil (3 + 2\sqrt{2})/\alpha \rceil$ , at least  $\lceil m/r \rceil$  signs can be chosen arbitrarily while every other sign remains fixed. Consequently there are at least  $2^{\lceil m/r \rceil}$  unordered good partitions. The argument removes an operator-small part given by the MSS partition theorem and applies the sharper Bownik–Casazza–Marcus–Speegle bound to its normalized complement. Prior work already provides counting estimates at universal-constant discrepancy scales; this note concerns the coordinate-subcube conclusion at the explicit MSS bound.

## 1 The quantitative statement

Let  $d, m \geq 1$  and put  $A_i = v_i v_i^*$ , where

$$v_i \in \mathbb{C}^d, \quad \sum_{i=1}^m A_i = I_d, \quad \|v_i\|^2 \leq \alpha > 0. \quad (1)$$

All norms of operators are spectral norms. Indices remain distinct even when vectors coincide or vanish. For  $\sigma \in \{-1, 1\}^m$  set

$$D_\sigma = \sum_i \sigma_i A_i, \quad N_A(t) = \#\{\sigma : \|D_\sigma\| \leq t\}.$$

An unordered indexed bipartition corresponds to the pair  $\{\sigma, -\sigma\}$ , so the number of good unordered partitions is  $N_A(t)/2$ . A coordinate subcube on a set  $F \subseteq [m]$  consists of all signings with prescribed signs on  $F^c$  and arbitrary signs on  $F$ .

Set

$$\tau(\alpha) = \sqrt{8\alpha} + 2\alpha, \quad \alpha_* = \frac{3 - 2\sqrt{2}}{2}.$$

The bound  $\tau$  follows from the two-part case of the MSS partition theorem [2]. The AIM workshop discussion [1] asks how many good partitions there are. It is an open-ended quantitative question; we fix the threshold and the counting convention above.

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\*Mercury Software GmbH. Email: [alper@mercurycodelab.com](mailto:alper@mercurycodelab.com). GitHub: [AlperTheKing](https://github.com/AlperTheKing).

**Theorem 1.** Suppose (1) holds and  $0 < \alpha < \alpha_*$ . Define

$$r = \left\lceil \frac{3 + 2\sqrt{2}}{\alpha} \right\rceil, \quad k = \left\lfloor \frac{m}{r} \right\rfloor. \quad (2)$$

There is a  $k$ -dimensional coordinate subcube on which  $\|D_\sigma\| < \tau(\alpha)$ . In particular,

$$N_A(\tau(\alpha)) \geq 2^{k+1}, \quad \frac{N_A(\tau(\alpha))}{2} \geq 2^k. \quad (3)$$

If  $\alpha \geq \alpha_*$ , all  $2^{m-1}$  unordered partitions are good.

For each fixed  $\alpha$  in the nontrivial range, (3) is exponential in  $m$ . It is neither a positive fraction of all signings uniformly in dimension nor an assertion of an optimal exponent.

Counting estimates already appear in the recent work of Akbas and Sra [4]. Their Corollary 4.4 gives at least  $2^m \exp(-Cd/K^2)$  signings at discrepancy  $K\sqrt{\alpha}$  for real positive semidefinite matrices satisfying  $\sum_i A_i = I_d$  and  $\max_i \text{tr } A_i \leq \alpha \leq 1$ , for  $K$  above a universal constant. Their introduction explicitly separates that scale from sharp MSS constants. Theorem 1 instead specifies the historical MSS bound and fixes the complement signs of an entire coordinate cube. We make no claim of a first exponential count, numerical dominance of all prior counts, or absolute priority for this elementary deduction.

## 2 Removing the free coordinates first

We use two previously established existence theorems. MSS [2, Corollary 1.5] gives, for every positive integer  $r$ , a partition  $[m] = F_1 \sqcup \dots \sqcup F_r$  such that

$$\left\| \sum_{i \in F_j} A_i \right\| \leq \left( \frac{1}{\sqrt{r}} + \sqrt{\alpha} \right)^2 \quad (1 \leq j \leq r). \quad (4)$$

Bownik, Casazza, Marcus and Speegle [3, Corollary 1.3 and its proof] show that a complex Parseval frame with squared norms at most  $a < 1/4$  has a signing with discrepancy at most

$$b(a) = \sqrt{8a(1-2a)}. \quad (5)$$

For clarity, rescale their vectors with parameter  $\eta = 1/a > 4$ . Their two frame operators, after dividing by  $\eta$ , have norm at most  $1/2 + \sqrt{2a(1-2a)}$ . Since they sum to the identity, their difference has norm at most twice the square-root term, giving (5).

**Lemma 2.** Suppose  $F \subseteq [m]$ ,  $M = \sum_{i \in F} A_i \leq \beta I_d$ ,  $0 \leq \beta < 1$ , and  $a = \alpha/(1-\beta) < 1/4$ . There is one fixed signing  $\varepsilon$  on  $F^c$  such that every extension to  $F$  has discrepancy at most

$$T(\alpha, \beta) = \beta + b\left(\frac{\alpha}{1-\beta}\right). \quad (6)$$

Moreover  $N_A(T(\alpha, \beta))/2 \geq 2^{|F|}$ .

*Proof.* Let  $B = I_d - M$ . Then  $(1-\beta)I_d \leq B \leq I_d$ , so  $B$  is positive definite. For  $i \notin F$ , put  $w_i = B^{-1/2}v_i$ . These vectors form a Parseval frame, and

$$\|w_i\|^2 = v_i^* B^{-1} v_i \leq \frac{\alpha}{1-\beta} = a.$$

Apply (5) to obtain a signing  $\varepsilon$  on  $F^c$ . If  $X = \sum_{i \notin F} \varepsilon_i w_i w_i^*$ , then

$$\left\| \sum_{i \notin F} \varepsilon_i A_i \right\| = \left\| B^{1/2} X B^{1/2} \right\| \leq \|B\| \|X\| \leq b(a).$$

For every signing  $\sigma$  on  $F$ , Loewner order gives

$$-M \leq \sum_{i \in F} \sigma_i A_i \leq M.$$

Thus the free signed sum has norm at most  $\beta$ , and the triangle inequality proves (6). The complement signing is the same for all  $2^{|F|}$  assignments.

Since  $B$  is positive definite and  $d \geq 1$ , the complement is nonempty. The resulting coordinate cube and its antipode are therefore disjoint. Both are good, and no two members of the original cube represent the same unordered partition. This proves the count.  $\square$

*Proof of Theorem 1.* If  $\alpha \geq \alpha_*$ , then  $\tau(\alpha) \geq 1$ . Every signing satisfies  $-I_d \leq D_\sigma \leq I_d$ , proving this case.

Assume  $0 < \alpha < \alpha_*$ . Choose an MSS  $r$ -partition with  $r$  as in (2), and let  $F$  be a largest part by cardinality. Then  $|F| \geq k$ . Since  $r\alpha \geq (1 + \sqrt{2})^2$ , (4) implies

$$\left\| \sum_{i \in F} A_i \right\| \leq \left( \frac{1}{\sqrt{r}} + \sqrt{\alpha} \right)^2 \leq 2\alpha.$$

Use Lemma 2 with  $\beta = 2\alpha$ . The hypotheses hold because  $\alpha < \alpha_* < 1/6$  implies  $a = \alpha/(1 - 2\alpha) < 1/4$ . A direct calculation gives

$$b(a)^2 = \frac{8\alpha(1 - 4\alpha)}{(1 - 2\alpha)^2}, \quad 8\alpha - b(a)^2 = \frac{32\alpha^3}{(1 - 2\alpha)^2} > 0. \quad (7)$$

Hence  $T(\alpha, 2\alpha) < 2\alpha + \sqrt{8\alpha} = \tau(\alpha)$ . If  $|F| > k$ , fix  $|F| - k$  of its signs. The resulting  $k$ -cube and its disjoint antipode prove (3).  $\square$

*Remark 3.* The order of operations matters. Flipping a set of signs in an existing full-frame signing costs at most twice the norm of its frame operator. Removing that set before constructing the complement witness reduces the cost to its norm. This is why the simple choice above permits  $r$  of order  $\alpha^{-1}$ . The proof is an existence argument; no efficient witness-finding algorithm is asserted here.

For example,  $\alpha = 1/12$  and  $m = 1200$  give  $r = 70$  and  $k = 18$ , hence at least 262144 unordered good partitions. For  $\alpha = 1/100$ , one has  $r = 583$ ; at  $m = 10^6$  the free dimension is at least 1716.

### 3 Counting does not imply positive density

An elementary repeated-coordinate frame illustrates a limitation that is already recorded in the upstream research report [5]. For integers  $q, d \geq 1$ , take  $q$  copies of each  $e_j/\sqrt{q}$ . Then  $m = qd$  and  $\alpha = 1/q$ . Define

$$g_q(t) = \sum_{\substack{0 \leq s \leq q \\ |2s - q| \leq qt}} \binom{q}{s}.$$

Counting the positive signs in each coordinate separately gives

$$N_A(t) = g_q(t)^d. \quad (8)$$

For  $t < 1$ ,  $g_q(t) \leq 2^q - 2$ , so the good fraction  $(g_q(t)/2^q)^d$  tends to zero with  $d$  and fixed  $q$ . In particular,  $q \geq 12$  places  $\tau(1/q)$  below 1.

There is a related bound on coordinate-cube geometry. If  $k_j$  free indices belong to coordinate  $j$ , its two extreme assignments have diagonal entries separated by  $2k_j/q$ . If the cube is  $t$ -good, both entries lie in  $[-t, t]$ , whence  $k_j \leq \lfloor qt \rfloor$ . Every such cube therefore has dimension at most  $d \lfloor qt \rfloor$ . This is only an upper bound; parity can make it nonsharp, and it does not establish optimality of Theorem 1.

The general proof above is analytic. Accompanying integer and rational regressions check the parameter inequalities, noncommuting two-dimensional examples, antipodal counting and (8). They do not prove the cited existence theorems or exhaust all complex frames. This unrefereed note was developed with AI assistance in derivation, source comparison, computation and exposition. It has been self-audited but has not received independent review or proof-assistant verification. Its contribution is the precise consequence in Theorem 1, not closure of the entire workshop agenda or a claim of sharp enumeration.

## References

- [1] American Institute of Mathematics, *Beyond Kadison–Singer: paving and consequences*, workshop discussion notes, 2014, Visontai and Edidin counting questions. [Primary discussion document](#).
- [2] A. W. Marcus, D. A. Spielman and N. Srivastava, *Interlacing Families II: Mixed Characteristic Polynomials and the Kadison–Singer Problem*, [arXiv:1306.3969v4](#), 2014.
- [3] M. Bownik, P. G. Casazza, A. W. Marcus and D. Speegle, *Improved bounds in Weaver and Feichtinger Conjectures*, [arXiv:1508.07353v2](#), 2016.
- [4] E. Akbas and S. Sra, *Boolean Small-Ball Inequalities for Discrepancy Theory*, [arXiv:2609.20785v1](#), 2026.
- [5] ulamai, *UnsolvedMath*, version 1.6.0, record AIM-ANALYSIS-0095 and accompanying research report, commit `c6e7f41b4eb6ce876feb83e7eb626be1b243fd7e`. [Hugging Face dataset](#).