Sharp Induced-Norm Paving for Symmetric Weighing Matrices

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Abstract

Let W be a real symmetric zero-diagonal matrix with entries in $\{0, \pm 1\}$ and $W^2 = dI$, where $d \geq 1$. For $1 \leq p \leq \infty$, write $q = \min(p, p')$, with $q = 1$ at the endpoints. We deduce from the multi-paving theorem of Ravichandran and Srivastava that W has a coordinate paving into at most $\lceil 36\varepsilon^{-q} \rceil$ parts whose compressions have induced ℓ_p norm at most $\varepsilon \|W\|_{p\to p}$. The exponent q is optimal uniformly over symmetric conference matrices. The elementary transfer estimate is $\|W[S]\|_{p\to p} \leq \|W[S]\|_{2\to 2}^{2/q}$: unit-modulus entries convert spectral control into a bound on surviving column degrees. A single spectral paving also gives quantitative estimates for all p simultaneously. This short, theorem-dependent note concerns a precise special case of induced-norm paving, not arbitrary zero-diagonal matrices. No absolute priority claim is made.

1 Statement and scope

All norms below are induced operator norms on finite-dimensional complex ℓ_p spaces, not Schatten norms; we abbreviate $\|A\|_{p\to p}$ by $\|A\|_p$. A coordinate paving of a matrix $A \in \mathbb{C}^{n\times n}$ is a partition π of $[n]$ into nonempty sets. For $S \in \pi$, the principal submatrix is denoted by $A[S]$. An ε-paving satisfies $\|A[S]\|_p \leq \varepsilon \|A\|_p$ for every part.

We consider real symmetric weighing matrices with zero diagonal:

$$W = W^{\mathsf{T}}, \qquad w_{ii} = 0, \qquad w_{ij} \in \{0, \pm 1\}, \qquad W^2 = dI, \qquad d \geq 1. \tag{1}$$

A symmetric conference matrix is the case $d = n - 1$, so every off-diagonal entry is 1 or -1. Let p' be conjugate to p, with $1' = \infty$ and $\infty' = 1$, and set $q(p) = \min(p, p')$.

Theorem 1. *For every matrix (1) and every $0 < \delta < 1$, there is one partition π with at most $\lceil 36\delta^{-2} \rceil$ parts such that*

$$\max_{S\in\pi} \|W[S]\|_p \leq \delta^{2/q(p)} \|W\|_p \qquad (1 \leq p \leq \infty). \tag{2}$$

In particular, for fixed p and $0 < \varepsilon < 1$, an ε-paving exists with at most $\lceil 36\varepsilon^{-q(p)} \rceil$ parts. Any number of parts that works uniformly for all symmetric conference matrices must be at least $\varepsilon^{-q(p)}$.

Thus the optimal uniform order of the number of parts for this class is $\Theta(\varepsilon^{-q(p)})$. The partition in (2) is independent of p. By contrast, the choice $\delta = \varepsilon^{q(p)/2}$ used for the sharper fixed-p count depends on p. We do not claim that one partition simultaneously attains all the different fixed-p counts. Taking $\delta = \varepsilon$ does give one relative-ε paving for every p, with at most $\lceil 36\varepsilon^{-2} \rceil$ parts.

The motivating question of Bill Johnson in the AIM workshop problem list [1] asks for dimension-independent paving of arbitrary zero-diagonal matrices in each fixed induced ℓ_p norm, for $1 \leq p < \infty$.

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That question does not impose symmetry, unit entries, or orthogonality. Theorem 1 does not resolve that general question. Its deep input is the published Hilbert-space multi-paving theorem [3]; the norm transfer and matching lower bound are elementary. The note is not a new proof of the Kadison–Singer theorem.

2 Two norm identities and estimates

Lemma 2. *For a matrix satisfying* (1),

$$\|W\|_p = d^{1/q(p)} \qquad (1 \le p \le \infty). \tag{3}$$

Proof. The diagonal entries of $W^2 = dI$ show that each row, and hence each column, has exactly d nonzero entries. Consequently $\|W\|_1 = d$, whereas symmetry and $W^2 = dI$ give $\|W\|_2 = \sqrt{d}$. For $1 \le p \le 2$, set $\theta = 2 - 2/p$. Riesz–Thorin interpolation yields

$$\|W\|_p \le d^{1-\theta}(\sqrt{d})^\theta = d^{1/p}.$$

Each coordinate vector e_j has norm one and $\|We_j\|_p = d^{1/p}$, which proves equality. For $2 \le p \le \infty$, duality and $W = W^*$ give $\|W\|_p = \|W\|_{p'}$. This also proves the endpoint assertion. ☐

Lemma 3. *Every principal submatrix $B = W[S]$ satisfies*

$$\|B\|_p \le \|B\|_2^{2/q(p)} \qquad (1 \le p \le \infty). \tag{4}$$

In fact, this estimate holds for every real symmetric matrix whose entries belong to $\{0, \pm 1\}$, without an orthogonality assumption.

Proof. Put $t = \|B\|_2$. The unit-modulus property gives, for each column,

$$\sum_i |b_{ij}| = \sum_i |b_{ij}|^2 = \|Be_j\|_2^2 \le t^2.$$

It follows that $\|B\|_1 \le t^2$. If $t = 0$, then $B = 0$ and the claim is immediate. Otherwise, interpolation between 1 and 2 gives

$$\|B\|_p \le (t^2)^{1-\theta} t^\theta = t^{2/p} \qquad (1 \le p \le 2), \qquad \theta = 2 - 2/p.$$

Symmetry and duality give the remaining exponents, including infinity. ☐

For a conference compression, every column degree is $|S| - 1$, so the proof also gives $|S| - 1 \le \|W[S]\|_2^2$. For sparse weighing matrices, the quantity controlled is the largest surviving column degree, not the cardinality of the part. No assertion that a compression satisfies a scalar-square identity is used.

3 Transfer of Hilbert multi-paving

We use the following fixed-part-count consequence of the proof of Ravichandran–Srivastava [3], Theorem 1. In the cited arXiv version, the relevant estimate occurs on pages 17–18. For $k \ge 2$ zero-diagonal Hermitian contractions $H_1, \dots, H_k$ and an integer $r \ge 1$, one can partition the coordinates into at most r parts with

$$\lambda_{\max}(H_j[S]) \le 3\sqrt{\frac{2k}{r}} \qquad (1 \le j \le k) \tag{5}$$

on every nonempty part. Empty parts may be discarded. For $r \ge n$ we can instead use singleton parts. We have weakened the source's strict inequalities to non-strict ones, so the rounding below causes no endpoint issue.

Proof of the upper bounds in Theorem 1. Apply (5) to the pair of zero-diagonal Hermitian contractions

$$H_1 = d^{-1/2}W, \qquad H_2 = -d^{-1/2}W, \qquad r = \lceil 36\delta^{-2} \rceil.$$

Both signs are needed: their simultaneous upper eigenvalue bounds give

$$\|W[S]\|_2 \le \delta\sqrt{d}$$

for every part. Lemmas 2 and 3 now imply

$$\|W[S]\|_p \le (\delta\sqrt{d})^{2/q(p)} = \delta^{2/q(p)} \|W\|_p .$$

The same partition works for every p. Substituting $\delta = \varepsilon^{q(p)/2}$ proves the fixed-p bound. □

4 Optimality on conference matrices

Lemma 4. *Let C be a symmetric conference matrix of order n. Any partition into at most r parts, where $n \ge r$, has a part S for which*

$$\frac{\|C[S]\|_p}{\|C\|_p} \ge \left(\frac{n/r - 1}{n-1}\right)^{1/q(p)}. \tag{6}$$

Proof. There is a part of size $s \ge n/r$. For $1 \le p \le 2$, testing a coordinate vector in this part gives $\|C[S]\|_p \ge (s-1)^{1/p}$. Duality gives $\|C[S]\|_p \ge (s-1)^{1/q(p)}$ for all p. Divide by $\|C\|_p = (n-1)^{1/q(p)}$ from Lemma 2. □

For clarity, an unbounded sequence of real symmetric conference matrices is supplied by the classical Paley construction [2]. We recall the short character calculation, so the lower bound does not depend on an unstated existence hypothesis. Let $Q = 5^a$, with $a \ge 1$, and let χ be the quadratic character of $\mathbb{F}_Q$, extended by $\chi(0) = 0$. Since $Q \equiv 1 \pmod 4$, one has $\chi(-1) = 1$. Define $M_{xy} = \chi(x-y)$ for $x, y \in \mathbb{F}_Q$. Then $M = M^{\mathsf{T}}$ and $M\mathbf{1} = 0$.

The character identity

$$\sum_{t \in \mathbb{F}_Q} \chi\big(t(t-1)\big) = -1 \tag{7}$$

follows by counting pairs (t, y) with $y^2 = t(t-1)$. Their number is Q plus the sum in (7). The invertible substitutions $u = 2t - 1$ and $v = 2y$ transform the equation into $(u-v)(u+v) = 1$, which has exactly $Q - 1$ solutions, parametrized by the nonzero value of $u - v$.

The diagonal entries of M^2 are $Q - 1$. For $x \ne y$, put $z = y + (x-y)t$ in the sum defining $(M^2)_{xy}$. Multiplicativity of χ and $\chi(-1) = 1$ reduce that sum to (7). Therefore $M^2 = QI - J$, where J is the all-ones matrix. It follows that

$$C = \begin{pmatrix} 0 & \mathbf{1}^{\mathsf{T}} \\ \mathbf{1} & M \end{pmatrix}, \qquad C^2 = QI,$$

is a symmetric conference matrix of order $Q + 1$.

Proof of the lower bound in Theorem 1. Fix p and suppose at most r parts suffice for a relative error ε uniformly over symmetric conference matrices. Apply Lemma 4 along the unbounded orders $n = 5^a + 1$. Passing to the limit gives $r^{-1/q(p)} \le \varepsilon$, equivalently $r \ge \varepsilon^{-q(p)}$. □

5 Provenance and reproducibility

The upper bound is a short deduction from [3], standard interpolation, and the unit-entry identity in Lemma 3. The lower bound uses the classical conference construction. These observations may overlap with folklore; the bounded literature search accompanying this note does not certify novelty or priority. The constant 36 is an available explicit upper constant, not a claimed optimal leading constant.

The accompanying source package includes the proof record, a hypothesis audit, and an exact-arithmetic regression script. The script checks finite Paley instances, all real symmetric zero-diagonal 4×4 matrices with entries in $\{0, \pm 1\}$, and interpolation identities, together with auxiliary diagnostics from the research record. Its 15,701 assertions are finite checks, not a proof for all orders or all real exponents. The mathematical argument is given above; neither formal verification nor independent peer review is claimed.

AI assistance was used in developing the argument, searching sources, preparing the manuscript, and running computational checks. This is an unrefereed preprint; responsibility for its claims rests with the author.

References

[1] American Institute of Mathematics, *Beyond Kadison–Singer: paving and consequences*, workshop problem list, question of Bill Johnson on induced ℓ_p paving. Public problem list. Accessed 9 September 2026.

[2] R. E. A. C. Paley, *On orthogonal matrices*, J. Math. Phys. **12** (1933), 311–320. doi:10.1002/sapm1933121311.

[3] M. Ravichandran and N. Srivastava, *Asymptotically optimal multi-paving*, Int. Math. Res. Not. **2021** (2021), no. 14, 10908–10940. doi:10.1093/imrn/rnz111. Theorem 1 and its proof in arXiv:1706.03737v2 (2017), pp. 1 and 17–18, are the version used here.