

Failure of Uniform Bounds for Fractional Carleson Operators at Exponent Zero

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Abstract

Let $\mathcal{C}_\varepsilon$ be the maximal singular integral with kernel $e^{i\lambda|t|^\varepsilon}/t$, where the supremum is over $\lambda \in \mathbb{R}$. We prove that $\|\mathcal{C}_\varepsilon\|_{L^2 \rightarrow L^2}$ grows at least as $\sqrt{\log(1/\varepsilon)}$ as $\varepsilon \downarrow 0$. More explicitly, the norm is at least $\sqrt{3N/(2048\pi)}$ whenever $N \geq 1$ and $0 < \varepsilon \leq 2^{-16-12N}$. This gives a negative answer to the even-phase uniformity question of Guo, also stated as Question 12 in the AIM workshop list *Carleson theorems and multilinear operators*. The proof uses packets centered at $\exp(16^j)$. A single modulation parameter, selected by finite base-16 digits, aligns their phases for the limiting logarithmic kernel. An explicit three-region estimate transfers the construction to positive exponents. The result does not concern the odd-phase uniformity question near exponent one.

1 The question and the lower bound

For $0 < \varepsilon < 1/2$, define

$$T_{\varepsilon,\lambda}f(x) = \text{p. v.} \int_{\mathbb{R}} f(x-t)e^{i\lambda|t|^\varepsilon} \frac{dt}{t}, \quad \mathcal{C}_\varepsilon f(x) = \sup_{\lambda \in \mathbb{R}} |T_{\varepsilon,\lambda}f(x)|. \quad (1)$$

We initially use smooth functions for which the paired integrals below converge. The norm in the theorem can equivalently be tested on $C_c^\infty(\mathbb{R})$.

Guo proves boundedness for each fixed nonexceptional exponent [2, Theorem 1.1]. His Question 1.4 asks, in particular, whether the even-phase estimates remain uniform near exponent zero. The positive-exponent portion is Question 12, equation (0.17), of the AIM workshop list [1]. Fixed-coefficient multiplier estimates are uniform in this regime [2, Theorem 1.6]; this does not settle the pointwise supremum over coefficients. Related fixed-exponent results appear in [3], while the local uniformity theorem in [4] concerns a different oscillatory Carleson family near exponent two.

Theorem 1. *For every integer $N \geq 1$ and every $0 < \varepsilon \leq 2^{-16-12N}$,*

$$\|\mathcal{C}_\varepsilon\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \geq \sqrt{\frac{3N}{2048\pi}}.$$

This lower bound holds when the input supremum is restricted to $C_c^\infty(\mathbb{R})$. It can be witnessed, up to approximation, using at most 16^N positive values of λ .

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Corollary 2. For $0 < \varepsilon \leq 2^{-56}$,

$$\|C_\varepsilon\|_{L^2 \rightarrow L^2} \geq \sqrt{\frac{\log_2(1/\varepsilon)}{16384\pi}}.$$

In particular there is no constant uniform over $0 < \varepsilon < 1/2$ in the estimate asked for in [1, Question 12].

Proof. Set $L = \log_2(1/\varepsilon)$ and $N = \lfloor (L - 16)/12 \rfloor$. Then $16 + 12N \leq L$, and for $L \geq 56$, $N \geq (L - 28)/12 \geq L/24$. Apply Theorem 1. $\square$

At a fixed coefficient, setting $\varepsilon = 0$ in the phase recovers a scalar multiple of the Hilbert transform. Our coefficient instead has size $1/\varepsilon$. After removing its constant phase, the relevant kernel is $|t|^{i\mu}/t$. The frozen UnsolvedMath research report for AIM-ANALYSIS-0071 already records this reduction and its multiplier [5]. We use them as background. The construction below supplies a finite-valued phase selector and a quantitative transfer estimate. The rate is a lower bound; no optimality or absolute priority claim is made.

2 The logarithmic multiplier

Our Fourier convention is

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx, \quad f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) e^{ix\xi} d\xi.$$

For real μ , write

$$S_\mu f(x) = \int_0^\infty [f(x-t) - f(x+t)] t^{i\mu} \frac{dt}{t}. \quad (2)$$

If f has bounded derivative and quadratic decay, this integral converges absolutely.

Lemma 3. On Schwartz functions, S_μ has multiplier

$$m_\mu(\xi) = a(\mu) \operatorname{sgn}(\xi) |\xi|^{-i\mu}, \quad a(\mu) = 2\Gamma(i\mu) \sinh(\pi\mu/2),$$

with $a(0) = -i\pi$. Moreover,

$$|a(\mu)|^2 = \frac{2\pi}{|\mu|} \tanh\left(\frac{\pi|\mu|}{2}\right), \quad |a(\mu)| \leq \pi, \quad (3)$$

and $|a(\mu)| \geq 1/2$ for $2\pi \leq \mu \leq 4\pi$.

Proof. The sine Mellin integral, initially evaluated for $0 < \operatorname{Re} s < 1$, extends to $-1 < \operatorname{Re} s < 1$: at zero use the sine factor, and at infinity integrate by parts. It gives

$$-2i \int_0^\infty t^{s-1} \sin(t\xi) dt = -2i\Gamma(s) \sin(\pi s/2) \operatorname{sgn}(\xi) |\xi|^{-s}.$$

At $s = i\mu$ this is the stated multiplier. The identity can be interpreted in tempered distributions by first inserting exponential damping and then passing to the limit on Schwartz test functions. The value at $\mu = 0$ follows by continuity.

The Gamma recurrence and reflection formulas give

$$|\Gamma(i\mu)|^2 = \frac{\pi}{|\mu| \sinh(\pi|\mu|)}.$$

This proves (3); its upper bound follows from $\tanh u \leq u$. For $2\pi \leq \mu \leq 4\pi$,

$$|a(\mu)|^2 \geq \frac{1}{2} \tanh(\pi^2) > \frac{1}{4}.$$

$\square$

3 Packets and a common phase selector

Fix $N \geq 1$, and put

$$B = 16, \quad \xi_j = e^{B^j} \quad (1 \leq j \leq N), \quad \phi(x) = \left(\frac{\sin(x/2)}{x/2} \right)^2, \quad f_N(x) = \phi(x) \sum_{j=1}^N e^{i\xi_j x}, \quad (4)$$

where $\phi(0) = 1$. The triangular transform identity gives

$$\widehat{\phi}(\eta) = 2\pi(1 - |\eta|)_+.$$

In particular,

$$0 \leq \phi \leq 1, \quad \phi(x) \leq 4/x^2 \quad (x \neq 0), \quad \phi(x) \geq 1/2 \quad (|x| \leq 1). \quad (5)$$

The last inequality follows from $\sin y/y \geq 1 - y^2/6$ for $|y| \leq 1/2$.

Plancherel's theorem gives $\|\phi\|_2^2 = 4\pi/3$. The translated Fourier supports $[\xi_j - 1, \xi_j + 1]$ are positive and pairwise disjoint. Hence

$$\|f_N\|_2^2 = \frac{4\pi N}{3}. \quad (6)$$

Also Fourier inversion gives $\|\phi'\|_\infty \leq 1/3$, so

$$\|f'_N\|_\infty \leq N/3 + \sum_{j=1}^N \xi_j \leq 2N\xi_N. \quad (7)$$

Lemma 4. *For every $x \in \mathbb{R}$ there is a measurable finite-valued choice $\mu(x) \in [2\pi, 17\pi/8]$ such that*

$$\operatorname{Re} e^{i\xi_j x - i\mu(x)B^j} \geq \cos(2\pi/B) > 1/2 \quad (1 \leq j \leq N).$$

The set of possible values has cardinality at most B^N .

Proof. With braces denoting fractional parts, set

$$\theta_j = \left\{ \frac{\xi_j x}{2\pi} \right\}, \quad d_j = \lfloor B\theta_j \rfloor, \quad \tau = \sum_{j=1}^N d_j B^{-(j+1)}, \quad \mu = 2\pi(1 + \tau).$$

Since $0 \leq d_j \leq B - 1$, we have $0 \leq \tau < 1/B$. For each j , the terms with index less than j in $B^j \tau$ are integers, and

$$\{B^j \tau\} = \frac{d_j}{B} + \sum_{k=j+1}^N d_k B^{j-k-1} \in [d_j/B, (d_j + 1)/B].$$

Thus θ_j and $\{B^j \tau\}$ differ by less than $1/B$. Because B^j is an integer, the extra 1 in $1 + \tau$ does not affect the exponential. This proves the phase estimate. There are B^N possible digit words, and their selection by floor functions is measurable. Half-open intervals cover boundary cases. $\square$

Lemma 5. *For the selector in Lemma 4,*

$$|S_{\mu(x)} f_N(x)| \geq N/16 \quad (|x| \leq 1).$$

Proof. For $|\eta| \leq 1$ and $2\pi \leq \mu \leq 4\pi$,

$$\left| (\xi_j + \eta)^{-i\mu} - \xi_j^{-i\mu} \right| \leq |\mu| |\log(1 + \eta/\xi_j)| \leq 8\pi/\xi_j.$$

Here $|\eta|/\xi_j < 1/2$. Since $\hat{\phi} \geq 0$, $\int \hat{\phi} = 2\pi$, and $|a(\mu)| \leq \pi$, the multiplier formula implies

$$S_\mu f_N(x) = a(\mu)\phi(x) \sum_{j=1}^N e^{i\xi_j x - i\mu B^j} + E_N(x, \mu), \quad |E_N| \leq 8\pi^2 \sum_{j=1}^N e^{-B^j}. \quad (8)$$

The use of the multiplier formula for these smooth non-Schwartz packets is justified in Section 5. Using $B^j \geq 16j$, $e > 2$, and $\pi < 4$, we obtain

$$|E_N| \leq \frac{8\pi^2}{e^{16} - 1} < \frac{128}{65535} < \frac{1}{16}.$$

On $[-1, 1]$, Lemmas 3 and 4, together with (5), now give

$$|S_{\mu(x)} f_N(x)| \geq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{N}{2} - \frac{1}{16} \geq \frac{N}{16}.$$

□

Notice that this also proves unboundedness on L^2 of the maximal logarithmic operator with μ restricted to a fixed compact interval. Each individual multiplier remains bounded.

4 Transfer to positive exponents

We prove Theorem 1. Let $h = B^N$, $a = e^{-2h}$, and $b = 64$, and suppose $\varepsilon h^3 \leq 2^{-16}$. For $|x| \leq 1$, set $g_x(t) = f_N(x - t) - f_N(x + t)$. As an equality of convergent paired integrals,

$$\begin{aligned} & e^{-i\mu/\varepsilon} T_{\varepsilon, \mu/\varepsilon} f_N(x) - S_\mu f_N(x) \\ &= \int_0^\infty \frac{g_x(t)}{t} \left[e^{i\mu(t^\varepsilon - 1)/\varepsilon} - e^{i\mu \log t} \right] dt. \end{aligned} \quad (9)$$

We estimate this uniformly for $2\pi \leq \mu \leq 4\pi$.

For $0 < t < a$, (7) gives $|g_x(t)| \leq 4Ne^{ht}$. Since the phase difference has modulus at most two, this part of (9) is bounded by

$$8Ne^h a = 8Ne^{-h} < N/128. \quad (10)$$

Indeed $h \geq 16$, and $8e^{-16} < 8/2^{16} < 1/128$.

For $t > b$, both $|x - t|$ and $|x + t|$ are at least $t/2$. By (5), $|g_x(t)| \leq 32N/t^2$. Thus this part is at most

$$64N \int_b^\infty t^{-3} dt = \frac{32N}{b^2} = \frac{N}{128}. \quad (11)$$

For $a \leq t \leq b$, we have $|\log t| \leq 2h$ and $\log(b/a) \leq 3h$. Taylor's theorem gives, for real u ,

$$\left| \frac{e^{\varepsilon u} - 1}{\varepsilon} - u \right| \leq \frac{\varepsilon u^2}{2} e^{\varepsilon|u|}.$$

Using $|g_x| \leq 2N$, and writing $M = 4\pi < 16$, the middle part of (9) is therefore at most

$$(2N)(3h) \frac{M\varepsilon}{2} (2h)^2 e^{2\varepsilon h} = 12MN\varepsilon h^3 e^{2\varepsilon h} \leq \frac{384N}{65536} = \frac{3N}{512} < \frac{N}{128}. \quad (12)$$

For the last inequality we used $2\varepsilon h \leq 2^{-23} < 1/2$, which implies $e^{2\varepsilon h} < 2$.

Adding (10)–(12) bounds the difference by $3N/128 < N/32$. Choose $\mu = \mu(x)$ as in Lemma 4. The scalar $e^{-i\mu/\varepsilon}$ has modulus one, so Lemma 5 implies

$$\mathcal{C}_\varepsilon f_N(x) \geq N/32 \quad (|x| \leq 1).$$

Consequently

$$\frac{\|\mathcal{C}_\varepsilon f_N\|_2^2}{\|f_N\|_2^2} \geq \frac{2(N/32)^2}{4\pi N/3} = \frac{3N}{2048\pi}.$$

The condition $\varepsilon h^3 \leq 2^{-16}$ is exactly $\varepsilon \leq 2^{-16-12N}$. The selector uses at most 16^N values, and all corresponding coefficients $\lambda = \mu/\varepsilon$ are positive. No infinite-supremum limit interchange is involved.

5 Test-function details and scope

For completeness, the multiplier identity used in (8) applies to f_N as follows. Convolve the triangular $\hat{\phi}$ with a nonnegative C_c^∞ approximate identity of mass one, and then translate to each ξ_j . The resulting Fourier functions are smooth and compactly supported in a fixed neighborhood of the positive packet supports. Their inverse transforms are Schwartz. In physical space they are f_N multiplied by smooth factors of modulus at most one, tending to one, whose first derivatives are uniformly bounded. Thus they have uniform derivative bounds and uniform quadratic decay. Their paired logarithmic integrals converge pointwise by domination at zero and infinity. Their Fourier transforms converge in L^1 on compact sets away from zero, so their multiplier expressions converge uniformly. Lemma 3 therefore gives the stated identity for f_N .

To recover compactly supported smooth inputs, fix $\chi \in C_c^\infty(\mathbb{R})$, with $0 \leq \chi \leq 1$ and $\chi = 1$ on $[-1, 1]$, and let $f_{N,R}(x) = \chi(x/R)f_N(x)$. Then $\|f_{N,R}\|_2 \rightarrow \|f_N\|_2$. On $|x| \leq 1$, the paired integrals for $T_{\varepsilon,\lambda}(f_{N,R} - f_N)$ vanish near the origin for $R > 2$; elsewhere their tails are bounded by a constant times Nt^{-3} , starting at $t \geq R - 1$. They consequently tend to zero uniformly on this interval and uniformly in λ . The lower ratio for f_N is therefore a limit of lower ratios for C_c^∞ inputs, completing the domain assertion in Theorem 1.

The conclusion disproves uniformity as the positive exponent tends to zero. It does not contradict boundedness for each fixed positive exponent, and it gives no answer to Guo’s odd-phase question near exponent one.

The accompanying verification material includes exact rational checks of the finite digit selector and constants, and numerical checks of the damped kernel and Gamma identities. These checks supplement the all- N analytic argument; they are not a formal verification or external review. AI assistance was used in developing the construction, checking the calculations, searching the literature, and preparing the text. This manuscript is an unrefereed preprint.

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