

Sharp Attraction Thresholds for Neutral Jordan Skew Products

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Abstract

For a class of holomorphic skew products tangent to the identity, we give necessary and sufficient conditions for an open attracting domain and for an open domain attracted along the base axis. The transverse leading coefficient is a fixed matrix with real spectrum and arbitrary Jordan blocks. If the base is $g(z) = z - z^{2t+1} + az^{3t+1} + O(z^{3t+2})$ and the fiber multiplier is $I - iz^t B - cz^{2t} I + R(z)$, with $R(z) = O(z^{2t+1})$ in the algebra generated by B , the attraction threshold is $\operatorname{Re} c > \max_{\beta} \{\beta^2/2 + \beta \operatorname{Im} a + t(m_{\beta} - 1)\}$. Attachment to the base axis requires the same strict inequality with one added to its right-hand side. Here m_{β} is the largest Jordan block size at β . We prove exact rates at every Jordan height, including the boundary cases, and construct invariant open domains. The criteria distinguish equal-spectrum examples and exhibit the effect of a higher base coefficient. These are explicit finite-jet criteria within the stated linear-fiber class, not a classification of general degenerate characteristic directions.

1 The family and the sharp criteria

Let $t \geq 1$ be an integer, put $r = 2t$, and let $a, c \in \mathbb{C}$. Let $B \neq 0$ be a complex $d \times d$ matrix whose eigenvalues are real. Consider the holomorphic germ

$$F(z, W) = (g(z), H(z)W), \tag{1}$$

$$g(z) = z - z^{r+1} + az^{r+t+1} + O(z^{r+t+2}), \tag{2}$$

$$H(z) = I - iz^t B - cz^r I + R(z), \quad R(z) = \sum_{j=0}^{d-1} r_j(z) B^j, \quad r_j(z) = O(z^{r+1}). \tag{3}$$

The missing intermediate coefficients in (2) are part of the hypotheses. The coefficients r_j are holomorphic near zero. In particular all fiber factors commute. No assertion about arbitrary noncommuting remainders or nonlinear fibers is intended.

A *local attracting domain* means a connected open set in a sufficiently small neighborhood of $(0, 0)$, with $(0, 0)$ on its boundary, which is forward invariant and whose points converge to $(0, 0)$. It is *attached to the base axis* if, for every point in the domain, the corresponding orbit satisfies $W_n/z_n \rightarrow 0$. This is first-order projective tangency, not infinite-order contact with a formal curve.

For an eigenvalue β of B , let m_{β} be its largest Jordan block size, and set

$$T_0 = \max_{\beta \in \operatorname{spec} B} \left\{ \frac{\beta^2}{2} + \beta \operatorname{Im} a + t(m_{\beta} - 1) \right\}, \quad T_1 = T_0 + 1. \tag{4}$$

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Theorem 1.1. *The germ (1) has a local attracting domain if and only if $\operatorname{Re} c > T_0$. It has one attached to the base axis if and only if $\operatorname{Re} c > T_1$. Both inequalities are strict. When an inequality holds, such a domain exists over each of the r attracting base directions. This does not assert a count of maximal connected basins.*

The derivative at the origin is the identity. The first nonlinear homogeneous term is $(0, -iz^t BW)$ and vanishes at $e_1 = (1, 0)$. Consequently $[e_1]$ is a degenerate characteristic direction. The fixed hyperplane $z = 0$ also shows that the origin is not an isolated fixed point. This is allowed in the broader question about degenerate directions in the AIM problem list [1].

Lapan's higher-degree theorem [2] supplies attracting domains under a strict transversal-attraction condition; the leading scalar coefficient considered here is neutral on the base rays. Lapan's examples [3] already demonstrate that higher-order terms can alter attraction, and her thesis [4, Chapter 4] discusses Jordan obstructions. The distinction between convergence to a point and convergence along a direction is therefore not new. Nor are shearing or irregular difference-equation asymptotics new methods: see, for example, Culmer–Harris [5] and Janas [6]. Our purpose is the explicit strict criterion (4), with all Jordan heights and an invariant-domain proof for the precise family (2)–(3). The proof is self-contained.

2 The base coordinate and its critical drift

Lemma 2.1. *For every sufficiently small nonzero forward base orbit converging to zero there is $\sigma \in \{1, -1\}$ such that*

$$z_n^t = \frac{\sigma}{\sqrt{r}} n^{-1/2} + \frac{a}{r} n^{-1} + O(n^{-1-1/r} + n^{-3/2} \log n), \quad (5)$$

$$\sum_{k < n} z_k^t = \frac{2\sigma}{\sqrt{r}} \sqrt{n} + \frac{a}{r} \log n + A_0 + o(1), \quad (6)$$

$$\sum_{k < n} z_k^r = \frac{1}{r} \log n + A_1 + o(1). \quad (7)$$

The constants depend on the orbit. Moreover, each of the r base directions has a sectorial holomorphic coordinate U with $U(g(z)) = U(z) + 1$. The expansion (5), with n replaced by $X = \operatorname{Re} U$, is uniform on inverse images of half-strips $X > X_0$, $|\operatorname{Im} U| < M$.

Proof. On a branch $z = \zeta(ru)^{-1/r}$, $\zeta^r = 1$, put $u = 1/(rz^r)$ and $\sigma = \zeta^t$. Expansion of (2) gives

$$G(u) := \frac{1}{rg(z)^r} = u + 1 - Au^{-1/2} + O(u^{-1/2-1/r}), \quad A = \frac{a\sigma}{\sqrt{r}}. \quad (8)$$

Terms of order u^{-1} are included in the displayed error, also when $r = 2$. The full expansion is convergent in powers of $u^{-1/r}$ near infinity on the branch.

Fix a narrow sector $S = \{\operatorname{Re} u > X_0, |\operatorname{Im} u| < \delta \operatorname{Re} u\}$. Increasing X_0 makes it forward invariant: $G(u) - u = 1 + o(1)$, so its real part increases by at least $1/2$, and on the two sloping boundaries the positive real displacement dominates the imaginary error.

For completeness, a translation coordinate can be built by a finite asymptotic correction followed by a convergent series. Choose

$$\Phi_0(u) = u + \sum_{j=1}^{r-1} b_j u^{1-j/r} + b_r \log u$$

so that $e(u) = \Phi_0(G(u)) - \Phi_0(u) - 1 = O(u^{-1-1/r})$. The choice is triangular: the new coefficient at order $u^{-j/r}$ is $(1 - j/r)b_j$ for $j < r$, and is b_r at order u^{-1} . Thus the nonzero denominators determine the coefficients successively. Equation (8) gives $b_1 = \dots = b_{t-1} = 0$ and $b_t = 2A$. The series

$$\Phi(u) = \Phi_0(u) + \sum_{n \geq 0} e(G^n(u))$$

converges normally on S , because $\operatorname{Re} G^n(u) \geq \operatorname{Re} u + n/2$. It satisfies $\Phi(G(u)) = \Phi(u) + 1$. On a smaller sector,

$$\Phi(u) = u + 2A\sqrt{u} + O(u^{1/2-1/r} + \log u), \quad \Phi'(u) = 1 + o(1). \quad (9)$$

The derivative estimate follows from Cauchy's estimate on discs of radius comparable to $|u|$ contained in the larger sector. On a sufficiently far right, smaller convex sector, $|\Phi' - 1| < 1/2$ proves injectivity by integration along segments. Rouché's theorem on discs about U of radius $K\sqrt{|U|}$, with fixed sufficiently large K , shows that the image contains every fixed-width right half-strip sufficiently far to the right. These discs stay in the smaller sector, and (9) bounds $|\Phi(u) - u|$ by a constant times $\sqrt{|U|}$ there.

Inverting (9) on such a half-strip gives

$$u = U - 2A\sqrt{U} + O(U^{1/2-1/r} + \log U).$$

It follows that

$$z^t = \frac{\sigma}{\sqrt{r}}U^{-1/2} + \frac{a}{r}U^{-1} + O(U^{-1-1/r} + U^{-3/2} \log U). \quad (10)$$

In particular the second coefficient is independent of the sign of the petal. This proves the uniform assertion.

For any converging nonzero base orbit, $|u_n| \rightarrow \infty$ and $u_{n+1} - u_n = 1 + o(1)$, hence $u_n/n \rightarrow 1$. Its branch is eventually fixed and it eventually enters the sector just used. There $U_n = U_0 + n$, so (10) proves (5). Subtracting integrals from the sums of $n^{-1/2}$ and n^{-1} gives constants and errors tending to zero. The remaining error in (5) is summable. Also $z_n^r = (rn)^{-1} + O(n^{-3/2})$, which proves (7). Finite initial segments only change the constants. $\square$

3 Every Jordan height and the equality cases

Put B into Jordan form. A nonzero vector w in a block $J = \beta I + N$ has *height* h if $N^h w = 0$ but $N^{h-1} w \neq 0$.

Theorem 3.1. *Fix any sufficiently small nonzero converging base orbit. The orbit of a height- h vector in a Jordan block satisfies*

$$\|w_n\| = K(z, w)n^{\kappa_\beta(h)}(1 + o(1)), \quad K(z, w) > 0, \quad \kappa_\beta(h) = \frac{\beta^2/2 + \beta \operatorname{Im} a - \operatorname{Re} c}{r} + \frac{h-1}{2}. \quad (11)$$

Thus it tends to zero exactly when $\kappa_\beta(h) < 0$, and it is $o(z_n)$ exactly when $\kappa_\beta(h) + 1/r < 0$.

Proof. All factors are invertible after the neighborhood is shrunk. Their logarithms are given by the convergent series at I and commute. On the block, with $x = z^t$,

$$\log H(z) = -ixJ + x^2(J^2/2 - cI) + O(|z|^{r+1} + |z|^{3t}).$$

The matrix remainder is a polynomial in N , and its norm is summable along a converging base orbit. Lemma 2.1 yields the product

$$P_n = \exp \left(-i\omega\sqrt{n}J + \frac{J^2/2 - cI - iaJ}{r} \log n + L + E_n \right), \quad \omega = \frac{2\sigma}{\sqrt{r}}, \quad E_n \rightarrow 0, \quad (12)$$

where L, E_n are polynomials in N . Extract the scalar factor

$$e^{-i\omega\beta\sqrt{n}n^{(\beta^2/2 - c - ia\beta)/r}}.$$

The remaining growing factor is

$$\exp \left\{ \left(-i\omega\sqrt{n} + \frac{\beta - ia}{r} \log n \right) N + \frac{\log n}{2r} N^2 \right\} e^{L+E_n}.$$

On a height- h vector its leading term is

$$\frac{(-i\omega)^{h-1}}{(h-1)!} n^{(h-1)/2} N^{h-1} e^L w.$$

Its coefficient is nonzero: e^L is invertible and commutes with N . Every other term has a smaller power of $\sqrt{n}$, possibly multiplied by a power of $\log n$. The factor e^{E_n} changes the leading coefficient by $o(1)$ and preserves height. Since β is real, the oscillatory scalar has modulus one. Taking norms proves (11). At either equality the appropriate scaled norm tends to a positive constant, rather than to zero. $\square$

The stable fiber over a base point is consequently the direct sum, over the Jordan blocks, of the kernel levels satisfying

$$\operatorname{Re} c > \beta^2/2 + \beta \operatorname{Im} a + t(h-1).$$

The tangential stable fiber is obtained by adding one to the right side. The zero vector is included in every block. These are exact linear subspaces, not only statements about generic vectors. Constant projections onto the separate blocks exclude cancellation between them.

4 Invariant domains and necessity

Pointwise rates alone do not construct a forward invariant open domain. For a block of size m , use the shear, defined only for $z \neq 0$,

$$w = D(z)v, \quad D(z) = \operatorname{diag}(1, z^t, \dots, z^{t(m-1)}).$$

Since $D(z)^{-1}ND(z) = z^tN$, equations (2)–(3) give

$$v_1 = \left[I - i\beta z^t I - z^r C_m + o(|z|^r) \right] v, \quad C_m = cI - t \operatorname{diag}(0, 1, \dots, m-1) + iN. \quad (13)$$

In particular the remainder in R stays higher order after the shear: a polynomial in J is a polynomial in N , and the shear multiplies N^j by z^{tj} , never by a negative power. The diagonal factor $D(g(z))^{-1}D(z)$ equals $I + tz^r \operatorname{diag}(0, \dots, m-1) + o(|z|^r)$.

Suppose first that $\operatorname{Re} c > T_0$. Choose $\gamma > 0$ so that, in every block,

$$\operatorname{Re} c > \beta^2/2 + \beta \operatorname{Im} a + t(m-1) + \gamma. \quad (14)$$

If $\operatorname{Re} c > T_1$, choose $\gamma > 1$ instead. Choose η with

$$\beta^2/2 + \beta \operatorname{Im} a + \gamma < \eta < \operatorname{Re} c - t(m-1).$$

There is a Hermitian norm in which $\operatorname{Re}\langle C_m v, v \rangle \geq \eta \|v\|^2$. Indeed, putting $A_m = C_m - \eta I$, the positive definite matrix

$$Q = \int_0^\infty e^{-A_m^* s} e^{-A_m s} ds$$

is finite because all eigenvalues of A_m have positive real parts. The identity $A_m^* Q + Q A_m = I$ gives the assertion in the Q -norm.

Use Lemma 2.1 on a fixed-width invariant half-strip in U . For $X = \operatorname{Re} U \rightarrow \infty$, (10) and (13) imply

$$\|v_1\| \leq \left[1 - \frac{\eta - \beta^2/2 - \beta \operatorname{Im} a}{rX} + o(X^{-1}) \right] \|v\|, \quad (15)$$

$$|g(z)/z|^\gamma = 1 - \frac{\gamma}{rX} + o(X^{-1}). \quad (16)$$

The term $\beta \operatorname{Im} a$ in (15) comes from the coefficient $a/(rX)$ in z^t ; the term $\beta^2/2$ comes from the squared modulus of the leading imaginary scalar. All errors are uniform on the half-strip. The strict gap therefore makes the factor in (15) smaller than that in (16) for large X .

Over the half-strip define, in every block, $\|v_{\text{block}}\| < \epsilon |z|^\gamma$. This set and its image in (z, W) are connected, open and forward invariant. Choosing the half-strip sufficiently far right and ϵ small keeps the image in any prescribed neighborhood. Since $|z| < 1$, the entries of $D(z)$ are bounded, so $\|W\| = O(|z|^\gamma)$. This proves attraction, and proves attachment when $\gamma > 1$. The origin is on the boundary. The same construction works on each of the r branches.

Conversely, let an arbitrary local attracting domain be given. Its projection to the base is open, and hence contains a nonzero base point. The corresponding vertical slice is a nonempty open subset of $\mathbb{C}^d$; it contains a vector of maximal height in every Jordan block. The factors are invertible and preserve heights. Theorem 3.1 therefore forces $\operatorname{Re} c > T_0$. If the domain is attached, it forces $\operatorname{Re} c > T_1$. The positive limiting constants in that theorem exclude both equality cases. This completes the proof of Theorem 1.1.

5 Examples, coordinate changes and scope

With $t = 1$, $a = 0$, $c = 2$, $R = 0$ and $g(z) = z - z^3$, compare

$$B_0 = I_2, \quad B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The spectra, with multiplicities, agree. For B_0 , $T_1 = 3/2$, so attachment holds. For B_1 , $T_0 = 3/2$ and $T_1 = 5/2$: there is open attraction to the origin but no open domain attached to the base axis. The generic block vector has norm of order $n^{-1/4}$, while the base has modulus of order $n^{-1/2}$. At $c = 5/2$ its ratio to the base has a positive limiting norm; at $c = 3$ it tends to zero.

The critical base coefficient has an effect even in a scalar fiber. For $t = 1$, $B = 1$, $c = 1$, $R = 0$, the two polynomial bases

$$g_0(z) = z - z^3, \quad g_1(z) = z - z^3 - iz^4$$

give germs with the same third-order jet. The first has $T_1 = 3/2$ and does not admit attachment. The second has $T_1 = 1/2$ and does. Their fiber rates are respectively $n^{-1/4}$ and $n^{-3/4}$.

The correction is consistent with a change of base coordinate. Writing $z = h(\xi) = \xi + k\xi^{t+1}$ gives

$$h^{-1} \circ g \circ h(\xi) = \xi - \xi^{2t+1} + (a - tk)\xi^{3t+1} + O(\xi^{3t+2}),$$

and the transformed fiber coefficient at order ξ^{2t} is $cI + itkB$. The logarithmic residue is unchanged:

$$\frac{B^2}{2} - (cI + itkB) - i(a - tk)B = \frac{B^2}{2} - cI - iaB.$$

This is an algebraic consistency check; the transformed second coefficient need not be scalar and is not silently substituted into the statement of Theorem 1.1.

For the invariant axis $\Gamma = \{W = 0\}$, infinite-order contact is more restrictive. Every nonzero fiber component has a finite power rate (11), so it fails $W_n = o(z_n^k)$ for some integer k . Thus the stable set with contact at every order with Γ consists exactly of zero fibers over the attracting base orbits, even when there is an open domain with first-order tangency. The formal-curve asymptoticity in López-Hernanz et al. [7, Section 2] uses contact at every order and should not be conflated with the attachment condition here.

The criteria give a complete answer within (2)–(3); they do not solve the general AIM question [1]. In particular no arbitrary nonlinear-fiber, noncommuting-coefficient or global-automorphism conclusion follows. This is an unrefereed, self-audited preprint. AI assistance was used for research, derivation, computational checks and preparation of the manuscript. No independent review, proof-assistant verification or absolute-priority certification is claimed.

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