

Exact Degrees for a Fixed-Double-Pole Rational Hénon Family

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Abstract

We give an all-parameter degree calculation for the birational maps $F(x, y) = (\beta/x + \gamma/x^2 - \delta y, x)$ over $\mathbb{C}$, with $\gamma\delta \neq 0$. Outside the nontrivial root-of-unity resonances of $-\delta^3$, the first dynamical degree is 2. At a resonance of order ℓ , two explicit rational generating functions describe the pure inverse-square and mixed cases, with the usual QRT exception at $\delta = 1$. The proof treats every finite zero by its local singularity pattern and retains the boundary contributions on generic coordinate lines. In the mixed case an analytic product chart proves the repeated cancellations: some parameters have constant singularity multiplicities, whereas others have linearly growing multiplicities but the same degree series. Classical degree sequences and delayed-confinement polynomials are explicit antecedents. The subject is this particular rational family, not the full dynamics of arbitrary quadratic rational Hénon maps. This manuscript is unrefereed.

1 Statement and antecedents

For $\beta \in \mathbb{C}$ and $\gamma, \delta \in \mathbb{C}^*$, let

$$F(x, y) = \left(\frac{\beta}{x} + \frac{\gamma}{x^2} - \delta y, x \right), \quad F^n = (R_n, R_{n-1}), \quad R_{-1} = y, \quad R_0 = x. \quad (1)$$

The inverse is $(x, y) \mapsto (y, (\beta/y + \gamma/y^2 - x)/\delta)$. Bidegrees are taken on $\mathbb{P}^1 \times \mathbb{P}^1$. Set $d_{-1} = 0$ and $D(t) = \sum_{n \geq 0} d_n t^n$. For $\ell \geq 2$, write

$$A_\ell(t) = \sum_{j=0}^{\ell-1} t^{2j}, \quad B_\ell(t) = \sum_{j=0}^{\ell-2} t^{2j}, \quad C_\ell(t) = A_\ell(t) - 2tB_\ell(t).$$

Theorem 1.1. *For every parameter in (1) and every $n \geq 0$, the bidegree of R_n is (d_n, d_{n-1}) , with D given as follows.*

1. *If $q = -\delta^3$ is not a root of unity of order greater than one, then*

$$D(t) = \frac{1}{(1-t^2)(1-2t)}, \quad d_n = \frac{2^{n+3} - 3 + (-1)^n}{6}. \quad (2)$$

2. *If q has exact order $\ell \geq 2$ and $\beta = 0$, then*

$$D(t) = \frac{A_\ell(t)}{(1-t^2)C_\ell(t)}. \quad (3)$$

The same formula, with $\ell = 2$, holds for every β when $\delta = 1$.

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3. If $\beta \neq 0$, $\delta \neq 1$, and q has exact order $\ell \geq 2$, then

$$D(t) = \frac{A_\ell(t)}{(1-t^2)(C_\ell(t) - t^{2\ell-1})}. \quad (4)$$

In particular the pullback matrix of F^n , for $n \geq 1$, is

$$\begin{pmatrix} d_n & d_{n-1} \\ d_{n-1} & d_{n-2} \end{pmatrix}.$$

Theorem 1.2. *The first dynamical degree is 2 in (2) and 1 at $\delta = 1$. In (3) it is 1 for $\ell = 2$ and, for $\ell \geq 3$, the unique root $\lambda_\ell \in (1, 2)$ of $C_\ell(u) = 0$. In (4) it is the unique root $\mu_\ell \in (1, 2)$ of*

$$uC_\ell(u) - 1 = 0. \quad (5)$$

Both root sequences are strictly increasing and tend to 2. Here μ_2 is the square of the positive root of $v^3 - v - 1$.

Translation of both coordinates by a gives the same conclusions for $p(x) = c + \beta/(x - a) + \gamma/(x - a)^2$ when $c = (1 + \delta)a$. This is precisely the condition that the double pole be fixed by $x \mapsto c - \delta x$. Translation preserves the two fiber classes. The calculation concerns a special family within the broad investigation of quadratic rational Hénon maps in AIM problem 2.2 [1]; it does not settle that program or classify its Fatou sets and invariant measures.

The method is the zero/pole-counting method of Halburd [5], with its singularity-confinement antecedents. The generic degree sequence already occurs in Ohta–Tamizhmani–Grammaticos–Ramani [6], and the QRT endpoint is classical. Ramani–Grammaticos [7, pp. 28–29] print 0, 1, 2, 5, 9, 17, 30, 54, 95, ... for a nonautonomous inverse-square equation: this is the mixed $\ell = 2$ sequence, shifted by one initial index. The polynomial structure in Theorem 1.2 is also anticipated by the delayed-confinement calculation [4, eqs. (93)–(95)], after changing the pole weight from four to two. We do not claim priority for these sequences or polynomial structures. Our purpose is a uniform parameter calculation with the exceptional coefficients and repeated cancellations justified explicitly.

For comparison with the nonautonomous literature, choose $s^2 = \delta$ and put $R_n = s^n w_n$. Then

$$w_{n+1} + w_{n-1} = \frac{\beta s^{-2n-1}}{w_n} + \frac{\gamma s^{-3n-1}}{w_n^2}.$$

This preserves every iterate's degree, but is not an autonomous conjugacy. A generic assertion about the two coefficient sequences does not by itself establish every root-of-unity specialization used below.

2 Generic lines and nonresonant singularities

Work over an algebraic closure K of $\mathbb{C}(\eta)$, with η transcendental over $\mathbb{C}$. First take $r_{-1} = \eta$, $r_0 = z$, and iterate (1) in $K(z)$. Birationality ensures that these are nonzero rational functions, rather than identically zero or infinite expressions. A finite point's first nonunit among r_n , $n \geq 0$, can only be a zero with a unit predecessor: until that event the denominators are units. We call this a birth, with its actual zero multiplicity.

Put $\zeta = -\delta$, $q = \zeta^3$, and $S_j(q) = 1 + q + \cdots + q^j$. If a birth has local value $u\tau^m + O(\tau^{m+1})$, where $u \neq 0$ and $m > 0$, induction gives

$$\begin{aligned} r_{r+2j} &= \zeta^j u \tau^m + O(\tau^{m+1}), \\ r_{r+2j+1} &= \frac{\gamma \zeta^{-2j} S_j(q)}{u^2} \tau^{-2m} + O(\tau^{-2m+1}), \end{aligned} \quad (6)$$

as long as the displayed pole coefficients are nonzero. Indeed the reciprocal terms at a double pole have positive orders at least $2m$, so the next zero's leading coefficient is multiplied by ζ . The next pole coefficient is $\gamma/(\zeta^{2j+2}u^2)$ plus ζ times the old one, which gives (6).

All $S_j(q)$ are nonzero exactly when q is not a nontrivial root of unity; at $q = 1$ they are $j + 1$. In that case the alternating zero and double-pole pattern persists indefinitely, with unchanged zero order. Every finite nonunit belongs to exactly one first-birth pattern.

At infinity, direct induction yields

$$r_{2j} = \zeta^j z + O(1), \quad r_{2j+1} = \zeta^{j+1} \eta + O(z^{-1}). \quad (7)$$

Thus infinity contributes a simple pole exactly at even times, and never a zero. The odd constants remain nonzero. If Z_n is the total birth multiplicity at time n , equal zero and pole degrees on $\mathbb{P}^1$ give

$$d_n = Z_n + Z_{n-2} + \cdots = 2(Z_{n-1} + Z_{n-3} + \cdots) + \mathbf{1}_{2|n}.$$

Consequently $d_n = 2d_{n-1} + \mathbf{1}_{2|n}$, proving (2) for this axis.

For the other partial degree take $r_{-1} = z$, $r_0 = \eta$. The same finite-point argument applies, but infinity now contributes its simple poles at odd times. The prescribed zero of r_{-1} at $z = 0$ is not a birth: its following value is the prescribed unit η . The degree series is therefore $tD(t)$. These generic-line calculations are precisely degrees over $\mathbb{C}(y)$ in x and over $\mathbb{C}(x)$ in y ; algebraic extension of the coefficient field does not change them.

3 Finite packets: the pure and QRT cases

Suppose q has exact order $\ell \geq 2$ and $\beta = 0$. At a birth write the current value as ϵ , of valuation $m > 0$, and the preceding value as a unit w . All $O(\epsilon^k)$ statements below are valuation bounds in the original local parameter; ϵ need not itself be a uniformizer. Up to the first cancellation, induction refines (6) to

$$\begin{aligned} r_{r+2j} &= \zeta^j \epsilon + O(\epsilon^4), \\ r_{r+2j+1} &= \gamma \zeta^{-2j} S_j(q) \epsilon^{-2} + \zeta^{j+1} w + O(\epsilon). \end{aligned} \quad (8)$$

The reciprocal square of a double pole has order $4m$, and the reciprocal square of $\zeta^j \epsilon + O(\epsilon^4)$ has no additional constant term. These facts justify both remainders and the transported regular value in (8). At $j = \ell - 1$ the leading pole cancels, leaving $\zeta^\ell w + O(\epsilon)$, a unit. The following term is a unit as well, with residue $\gamma/(\zeta^\ell w)^2$.

The packet therefore has ℓ zeros of order m and $\ell - 1$ poles of order $2m$, at successive even and odd offsets. After it ends, any later first nonunit is again a zero with a unit predecessor. A point may be used for more than one packet; we count every chronological birth, with its own multiplicity. The packets are disjoint and exhaust all finite zeros and poles. With Z their birth series, (7) gives

$$D = A_\ell Z = 2tB_\ell Z + \frac{1}{1-t^2}.$$

This proves (3). The odd boundary correction on the second axis again gives tD .

For $\delta = 1$ and arbitrary β , expansion at a birth gives

$$\begin{aligned} r_{r+1} &= \gamma\epsilon^{-2} + \beta\epsilon^{-1} - w, \\ r_{r+2} &= -\epsilon + (\beta/\gamma)\epsilon^2 - (\beta^2/\gamma^2)\epsilon^3 + O(\epsilon^4), \\ r_{r+3} &= w + O(\epsilon). \end{aligned}$$

The double and simple poles and the additional constant contributions cancel in the last expression. This packet has two zeros and one double pole. Since its last term is a unit and the preceding term a zero, the next nonunit is again a zero; it may occur immediately at $r + 4$. Counting such nonoverlapping packets proves

$$D(t) = \frac{1+t^2}{(1-t^2)(1-t)^2}, \quad d_n = \left\lceil \frac{(n+1)^2}{2} \right\rceil.$$

The invariant $x^2y^2 - \beta xy - \gamma(x+y)$ is consistent with the known QRT endpoint; no new integrability assertion is needed.

4 The surviving simple pole

It remains to assume $\beta \neq 0$, $\delta \neq 1$ and q of exact order $\ell \geq 2$. Put $h = \zeta^\ell$, so $h^3 = 1$. Start a local block with

$$r_0 = u\epsilon + v\epsilon^2 + O(\epsilon^3), \quad r_{-1} = B\epsilon^{-1} + O(1), \quad u \neq 0.$$

At a first birth $B = 0$. The double-pole coefficient at offset $2\ell - 1$ cancels. Its surviving simple-pole coefficient is

$$B_{\text{new}} = \frac{\beta}{u}\sigma + hB, \quad \sigma = \zeta^{\ell-2}(\zeta + 2) \sum_{j=0}^{\ell-1} \zeta^{-2j} - \frac{2}{h}. \quad (9)$$

Here is a derivation that keeps the necessary subleading terms. For the contribution of β , normalize $\beta = \gamma = u = 1$, $v = B = 0$. The second coefficient c_j of the j th even zero is

$$c_j = \zeta^{j-1} \sum_{k=0}^{j-1} \frac{\zeta^k}{S_k(q)}.$$

The simple-pole coefficients satisfy $B_j = \zeta^{-j} - 2c_j\zeta^{-3j} + \zeta B_{j-1}$. Sum at $j = \ell - 1$ and interchange the finite sums, using

$$\sum_{j=k+1}^{\ell-1} q^{-j} = -q^{-k} S_k(q).$$

This gives (9). The initial B contributes hB . The initial v contributes the derivative in u of the cancelled double-pole coefficient, hence contributes zero. Restoring scales gives the displayed β/u factor. Only S_k with $k < \ell - 1$ were inverted.

If $B_{\text{new}} \neq 0$, the next zero has first coefficient $u_{\text{new}} = hu + \beta/B_{\text{new}}$. With $b = Bu/\beta$, one block of 2ℓ steps has boundary map

$$P = \sigma + hb, \quad \frac{u_{\text{new}}}{u} = h + \frac{1}{P}, \quad b_{\text{new}} = h^2b + 1 + h\sigma. \quad (10)$$

Future zero coefficients may vanish, so this boundary calculation alone does not justify repeating a fixed-multiplicity pattern.

Lemma 4.1. *If $h = 1$ then $\sigma = -2$. If h is a primitive cube root, then $\sigma \neq 0$, $\sigma \neq 1$ and $1 + h\sigma \neq 0$.*

Proof. Under our current assumptions $\zeta^2 \neq 1$. Summation in (9) gives

$$\sigma = \frac{h(1-h)(\zeta+2)}{\zeta^2-1} - 2h^2. \quad (11)$$

The case $h = 1$ follows. Suppose $h^2 + h + 1 = 0$. If $\sigma = 0$, then $2\zeta^2 + (h+2)\zeta + 2h + 2 = 0$. Its norm over $\mathbb{Q}(h)$ gives

$$4\zeta^4 + 6\zeta^3 + 7\zeta^2 + 6\zeta + 4 = 0.$$

Thus $s = \zeta + \zeta^{-1}$ satisfies $4s^2 + 6s - 1 = 0$. This irreducible quadratic has nonintegral monic coefficients, so its roots are not algebraic integers. But ζ is a root of unity, so s is an algebraic integer, a contradiction. Merely assuming $|\zeta| = 1$ would not prove this exclusion.

The condition $\sigma = 1$ reduces to $(2h+1)(\zeta^2 + \zeta + 1) = 0$, forcing $q = 1$, which is excluded. Finally $1 + h\sigma = 0$ would give

$$f(\zeta) = \zeta^2 + (h+2)\zeta + 2h + 3 = 0.$$

Since $|\zeta| = 1$, conjugate and multiply by ζ^2 to obtain $g(\zeta) = (1-2h)\zeta^2 + (1-h)\zeta + 1 = 0$. The Sylvester determinant of f, g is 27, after reduction by $h^2 + h + 1 = 0$. They have no common root. $\square$

5 Constant and growing multiplicities

If $h = 1$, (10) gives $b_{j+1} = b_j - 1$. At a birth set $u_0 = 1, b_0 = 0$. Then $b_j = -j$, $P_j = -(j+2)$ and $u_j = 1/(j+1)$. All necessary coefficients are nonzero in characteristic zero. Every 2ℓ -step block therefore has ℓ zeros of the original order m , $\ell - 1$ double poles of order $2m$ and a final pole of order m . Its zero and pole weight series are

$$\mathcal{A}(t) = \frac{A_\ell(t)}{1-t^{2\ell}}, \quad \mathcal{P}(t) = \frac{2tB_\ell(t) + t^{2\ell-1}}{1-t^{2\ell}}. \quad (12)$$

Now let h be primitive. Use the product chart $X = \text{current}$ and $S = X \text{ previous}$.

Lemma 5.1. *Near $(X, S) = (0, 0)$ the 6ℓ -step return is analytic and has the form*

$$X_{\text{new}} = XR(S/\beta) + X^2a(X, S), \quad S_{\text{new}} = S + Xb(X, S), \quad (13)$$

where a, b are analytic, $R(0) = 0$ and $R'(0) \neq 0$.

Proof. At the start of a block write the zero as $XU(X, S)$ and its predecessor as $X^{-1}V(X, S)$, with U a unit and V analytic. Intermediate zeros are X times units and double poles are X^{-2} times units, by the nonvanishing $S_j(q)$. At the last double pole its numerator vanishes identically along $X = 0$, for every S , and is divisible by X in the local analytic ring. The remaining term is $X^{-1}V_{\text{new}}$, where

$$V_{\text{new}}(0, S) = \beta\sigma/U(0, S) + hV(0, S).$$

If this is nonzero at $S = 0$ it is a unit, and the following zero is X times an analytic function. Its boundary value is (10). Derivatives of U in X have the vanishing contribution already accounted for in (9).

Start with $U = 1, V = S$, and put $b_0 = S/\beta$, $c = 1 + h\sigma$. The affine boundary map satisfies $b_3 = b_0$. At $b_0 = 0$ the subsequent b values are $c, (1 + h^2)c, 0$; the three pole coefficients P_j are

$$\sigma, \quad h(1 - \sigma), \quad -h^2.$$

They are nonzero by Lemma 4.1. The first two new zero coefficients are also nonzero. Their three-block product is

$$R(b_0) = \prod_{j=0}^2 \frac{b_{j+1}}{P_j}, \quad R'(0) = \frac{hc^2}{\sigma(1 - \sigma)} \neq 0.$$

The final zero is not inverted within these 6ℓ steps. Thus the chart construction has used only legitimate unit denominators. The last pole is X^{-1} times a unit V_3 and the boundary identity $U_3(0, S)V_3(0, S) = \beta b_3 = S$ proves the second part of (13). In particular $R'(0)V_3(0, 0) = \beta$. $\square$

We justify the genericity needed at the first return, rather than choosing a seemingly generic numerical predecessor. A finite zero on either generic coordinate line comes from a prime divisor E dominant over that line's parameter. Before its first zero, every current coordinate is finite and nonzero at the generic point of E . Each step is then a local isomorphism: F is regular for nonzero first coordinate and its inverse is regular for nonzero second coordinate. The image of E at its first zero is therefore dominant on the zero-coordinate line, not a point. Its predecessor w is nonconstant on E , and its residue in the algebraic closure of $\mathbb{C}(\eta)$ is transcendental over $\mathbb{C}$. It cannot equal a prescribed complex constant.

At that birth set $X = \epsilon$, $S = w\epsilon$, with $\text{ord } \epsilon = m$. Write $k = R'(0)/\beta \neq 0$ and $a_0 = a(0, 0)$. Lemma 5.1 gives

$$X_{\text{new}} = (kw + a_0)\epsilon^2 + O(\epsilon^3), \quad S_{\text{new}} = V_3(0, 0)(kw + a_0)\epsilon + O(\epsilon^2).$$

The displayed coefficients are units by the preceding argument. The new orders are thus $2m, m$. At subsequent returns, if $\text{ord } X = (j + 1)m$ and $\text{ord } S = m$, $j \geq 1$, the terms kXS and S in (13) dominate the remainders. The orders become $(j + 2)m, m$. Inside each return, the three 2ℓ -step blocks have the same multiplicity $(j + 1)m$. The zero and pole series are

$$\mathcal{A}(t) = \frac{A_\ell(t)}{(1 - t^{2\ell})(1 - t^{6\ell})}, \quad \mathcal{P}(t) = \frac{2tB_\ell(t) + t^{2\ell-1}}{(1 - t^{2\ell})(1 - t^{6\ell})}. \quad (14)$$

This proves the growing pattern for all times; finite agreement of several local expansions would not suffice.

In both mixed cases a birth never exits its singularity pattern, so every finite zero and pole belongs to exactly one first birth. The boundary behavior is still (7). If Z is the birth series, then $D = \mathcal{A}Z = \mathcal{P}Z + (1 - t^2)^{-1}$. Although (12) and (14) are different, they give the same ratio

$$\frac{\mathcal{P}}{\mathcal{A}} = \frac{2tB_\ell + t^{2\ell-1}}{A_\ell}.$$

Eliminating Z proves (4). All operations are formal power-series operations with finite sums at each coefficient. On the second axis the correction is $t/(1 - t^2)$, giving tD as before. This completes Theorem 1.1.

6 The positive growth root

The quadratic formula at $\ell = 2$ in the pure case gives dynamical degree one, and (2) gives two. For $\ell \geq 3$, C_ℓ is reciprocal and

$$(u^2 - 1)C_\ell(u) = u^{2\ell-1}(u - 2) + 2u - 1. \quad (15)$$

It changes sign between one and two. Put $m = 2\ell - 1$. On $(1, 2)$, zeros of (15) are zeros of $H(u) = m \log u + \log(2 - u) - \log(2u - 1)$. The sign of H' is that of $-2mu^2 + (5m - 3)u - 2m$. This concave quadratic has one zero in $(0, 1)$ and one in $(1, 2)$, since its values at $0, 1, 2$ have signs $-, +, -$. Thus H first increases and then decreases from its initial value zero to $-\infty$, with exactly one further zero. It is simple, and there are no positive zeros at $u \geq 2$.

For the mixed case put $T_\ell(u) = uC_\ell(u) - 1$. Then

$$(u^2 - 1)T_\ell(u) = u^{2\ell}(u - 2) + u^2 - u + 1. \quad (16)$$

Here $T_\ell(1) = 1 - \ell < 0$, $T_\ell(2) = 1$, and there are no zeros above two. On $(1, 2)$ use $H(u) = 2\ell \log u + \log(2 - u) - \log(u^2 - u + 1)$. Writing $x = u - 1$, the sign of its derivative is that of

$$2\ell - 2 - 4x - x^2 - (2\ell - 1)x^3.$$

This is strictly decreasing from a positive to a negative value on $(0, 1)$. The same argument proves the unique simple root μ_ℓ .

Reciprocity of C_ℓ identifies the first positive pole of the appropriate degree series as $1/\lambda_\ell$ or $1/\mu_\ell$. The numerator A_ℓ is positive there, so the pole does not cancel. All degree coefficients are nonnegative by the counting proof. Pringsheim's theorem [3] therefore identifies this positive pole with the convergence radius: an uncanceled complex pole of smaller modulus would force a smaller positive real singularity, which we have excluded. This gives the corresponding $\limsup d_n^{1/n}$. The ample degree is $d_n + 2d_{n-1} + d_{n-2}$; its root limit exists for birational surface maps [2], so its limit is the same number.

Finally

$$C_{\ell+1} = u^2 C_\ell + 1 - 2u, \quad T_{\ell+1} = u^2 T_\ell - (u^2 - u + 1)$$

show strict increase of the roots. Equations (15) and (16) force each increasing sequence to tend to two. At $\ell = 2$, $T_2 = u^3 - 2u^2 + u - 1$ vanishes at the square of the positive root of $v^3 - v - 1$, as polynomial reduction verifies. This proves Theorem 1.2.

7 Checks and scope

Exact reduced-rational regressions for the mixed cases test 160 partial degrees over $\mathbb{Q}$, $\mathbb{F}_{1009}$ and $\mathbb{F}_{1021}$, including both axes, both types of h and several coefficient choices. Precision-tracked Laurent arithmetic tests 456 orders and leading coefficients at cutoffs 192 and 320. Separate algebra and counting tests cover $\ell = 2, \dots, 20$ through index 100. These are reproducibility checks, not substitutes for the all-parameter proof. No all-time positive-characteristic statement is asserted.

The result does not classify nonfixed double poles or arbitrary two-pole collision strata. Its overlap with [6, 7, 4] is part of the statement's context, not evidence of a new general integrability criterion. In particular two different multiplicity patterns can have the same degree series. This note is a self-audited, unrefereed family calculation; it claims neither independent verification nor absolute priority. AI assistance was used in the derivations, literature comparison, symbolic regression programs and preparation of the text.

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