

Finite-Dimensional L^p -Bergman Spaces on Complete Reinhardt Domains

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Abstract

Coordinatewise completeness imposes an arithmetic restriction on the dimension of unweighted L^p -Bergman spaces. We show that, for every positive rational p , such a space on a complete Reinhardt domain in any complex dimension is either zero or infinite-dimensional. In $\mathbb{C}^2$, the only possible finite positive dimension is one, and it occurs for every irrational $p > 0$. An explicit unbounded domain with smooth boundary has a one-dimensional space for a countable dense set of irrational exponents and a zero space for all other positive exponents. Its ordinary Bergman space is zero, although its logarithmic image contains no affine line. For each irrational p and every integer $k \geq 2$, a related complete Reinhardt domain in $\mathbb{C}^3$ has an exactly k -dimensional L^p -Bergman space. The proofs use contained polydiscs, Taylor coefficients and explicit radial integrals; no pseudoconvexity is assumed.

1 Completeness and the dimension restriction

For a domain $\Omega \subset \mathbb{C}^d$ and $0 < p < \infty$, write

$$A^p(\Omega) = \left\{ f \in \mathcal{O}(\Omega) : \int_{\Omega} |f|^p dV < \infty \right\},$$

where dV is ordinary Lebesgue measure. When $p < 1$ this is a quasi-normed vector space, not a Banach space. All dimensions below are complex vector-space dimensions.

A domain is *complete Reinhardt* if

$$z \in \Omega, \quad |\lambda_j| \leq 1 \quad (1 \leq j \leq d) \quad \implies \quad (\lambda_1 z_1, \dots, \lambda_d z_d) \in \Omega. \quad (1)$$

This condition is stronger than invariance under coordinate rotations, and does not require pseudoconvexity. Wiergerinck's finite-dimensional examples concern Reinhardt domains [1]. In the pseudoconvex Reinhardt class, the ordinary Bergman-space dichotomy follows from logarithmic convex geometry; see Zwonek [2] and Jucha [3]. Engliš [6, Theorem 7] gives a related bounded-monomial argument from logarithmic convexity. Here contained polydiscs replace that convexity assumption, and the arithmetic of p becomes decisive.

Theorem 1.1. *Let Ω be a complete Reinhardt domain.*

1. *In any dimension, if $p > 0$ is rational, then $\dim A^p(\Omega)$ is zero or infinite.*
2. *If $\Omega \subset \mathbb{C}^2$ and $p > 0$, finite dimension of $A^p(\Omega)$ is at most one.*

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3. For a fixed $p > 0$, the possible dimensions in $\mathbb{C}^2$ are

$$\begin{cases} \{0, \infty\}, & p \in \mathbb{Q}, \\ \{0, 1, \infty\}, & p \notin \mathbb{Q}. \end{cases}$$

Here and below ∞ means infinite-dimensional, without specifying a Hamel cardinal. We first prove the restrictions; Section 2 supplies the examples realizing dimension one.

Lemma 1.2. *For a complete Reinhardt domain and $p > 0$, every nonzero Taylor coefficient of a function in $A^p(\Omega)$ corresponds to a monomial in $A^p(\Omega)$. Hence, writing*

$$S_p(\Omega) = \{\alpha \in \mathbb{N}_0^d : z^\alpha \in A^p(\Omega)\},$$

the space is finite-dimensional exactly when $S_p(\Omega)$ is finite, and in that case it is the span of those monomials.

Proof. Completeness and openness imply that $0 \in \Omega$ and that each point lies in an origin-centered polydisc whose closure is contained in Ω . Thus the Taylor expansion $f(z) = \sum c_\alpha z^\alpha$ converges throughout Ω . Fix a positive polyradius r whose closed polydisc is contained in Ω . Since $|f|^p$ is separately subharmonic, iterating the Poisson inequality in the coordinates gives

$$\sup_{|z_j| \leq r_j/2} |f(z)|^p \leq 3^d \int_{[0, 2\pi]^d} |f(r_1 e^{i\theta_1}, \dots, r_d e^{i\theta_d})|^p \frac{d\theta}{(2\pi)^d}.$$

The factor 3 bounds the Poisson kernel at modulus at most $1/2$. Cauchy's coefficient estimate on the half-sized polydisc therefore yields

$$|c_\alpha|^p r^{p\alpha} \leq 2^{p|\alpha|} 3^d \int_{[0, 2\pi]^d} |f(re^{i\theta})|^p \frac{d\theta}{(2\pi)^d}.$$

Integrating over the radial image of Ω against $(2\pi)^d \prod_j r_j dr_j$ proves

$$|c_\alpha|^p \int_\Omega |z^\alpha|^p dV \leq 2^{p|\alpha|} 3^d \int_\Omega |f|^p dV.$$

This proves the support assertion for all $p > 0$, including $p < 1$. If the support set is finite, every such Taylor series is a polynomial in the stated span. If it is infinite, the integrable monomials are an infinite linearly independent family. $\square$

For $p \geq 1$ the coefficient estimate also follows directly from torus integration and Jensen's inequality. Monomial bases in the Banach range are classical; a stronger norm-convergence result is given by Chakrabarti and Edholm [4]. We only need the support statement, whose proof above also covers the quasi-Banach range.

Lemma 1.3. *If $z^\alpha \in A^p(\Omega)$, put $q_\alpha = p\alpha + 2\mathbf{1}$. Then the real monomial $|z|^{q_\alpha}$ is bounded on Ω . If the positive cone generated by such vectors contains a nonzero $\gamma \in \mathbb{N}_0^d$, then $A^p(\Omega)$ is infinite-dimensional.*

Proof. For z with all coordinates nonzero, the polydisc of polyradius $|z|$ lies in Ω . Its monomial integral is

$$\frac{(2\pi)^d}{\prod_j (p\alpha_j + 2)} \prod_j |z_j|^{p\alpha_j + 2} \leq \int_\Omega |z^\alpha|^p dV. \quad (2)$$

Thus $|z|^{q_\alpha}$ is bounded. Products of positive real powers of these bounds show that z^γ is bounded whenever γ is as stated. Its integer exponents make it a single-valued holomorphic function, and the bound extends across coordinate axes. For one integrable z^α , every $z^{\alpha+k\gamma}$, $k \geq 0$, is then integrable. These are distinct linearly independent monomials. $\square$

Proof of the restrictions in Theorem 1.1. If $p = a/b$ with positive integers a, b and $A^p(\Omega) \neq \{0\}$, Lemma 1.2 supplies an integrable monomial z^α . The vector

$$\gamma = bq_\alpha = a\alpha + 2b\mathbf{1}$$

is a positive integer vector, so Lemma 1.3 proves the first assertion.

Now let p be irrational and $d = 2$. Suppose that $S_p(\Omega)$ contains distinct indices α, β . If q_α, q_β are linearly independent, their positive cone has nonempty interior in the positive quadrant. It contains a positive rational vector and, after scaling, a positive integer vector. Lemma 1.3 again gives infinite dimension.

If the two vectors are dependent, expansion of their determinant gives

$$p \det(\alpha, \beta) + 2(\alpha_1 - \alpha_2 - \beta_1 + \beta_2) = 0.$$

The coefficients are integers, so irrationality forces both to be zero. Consequently $\beta - \alpha = (t, t)$ for a nonzero integer t , and $0 = \det(\alpha, \beta) = t(\alpha_1 - \alpha_2)$. Both indices are diagonal, and q_α is a positive multiple of $(1, 1)$. This also supplies a bounded monomial, $z_1 z_2$, and infinite dimension. Therefore two distinct integrable monomials are incompatible with finite dimension. Lemma 1.2 finishes the second assertion. $\square$

2 A dense exceptional exponent spectrum

For a fixed irrational $a > 1$, define

$$F_a(r) = \frac{r^{-a}}{4 + (\log r)^2}, \quad \Omega_a = (\{0\} \times \mathbb{C}) \cup \{(z, w) : z \neq 0, |w| < F_a(|z|)\}. \quad (3)$$

Theorem 2.1. *The domain Ω_a is unbounded, complete Reinhardt and has C^∞ boundary. Set*

$$E_a = \left\{ \frac{2(a-1)}{m-an} : m, n \in \mathbb{N}_0, m > an \right\}. \quad (4)$$

This is a countable dense subset of $(0, \infty)$ containing no rational number. For every $p > 0$,

$$A^p(\Omega_a) = \begin{cases} \{0\}, & p \notin E_a, \\ \text{span}\{z^m w^n\}, & p = 2(a-1)/(m-an) \in E_a, \end{cases}$$

where the pair (m, n) in the second line is unique.

Proof. Writing $x = \log r$, we have

$$\frac{d}{dx} \log F_a(e^x) = -a - \frac{2x}{4+x^2} \leq -a + \frac{1}{2} < 0.$$

Thus decreasing $|z|$ enlarges the w -fiber, and decreasing $|w|$ preserves membership. Moreover $F_a(r) \rightarrow \infty$ as $r \downarrow 0$. Every point of the added w -axis has an open product neighborhood in Ω_a . The set is open and complete and is connected by coordinate contractions to the origin. Both

coordinate axes are interior, so every finite boundary point has $z, w \neq 0$. There $|w|^2 - F_a(|z|)^2$ is a smooth defining function with nonzero gradient in w . This proves the boundary assertion.

For nonnegative integers m, n , polar coordinates and integration first over the w -disc give

$$\int_{\Omega_a} |z^m w^n|^p dV = \frac{(2\pi)^2}{pn+2} \int_{\mathbb{R}} e^{[pm+2-a(pn+2)]x} (4+x^2)^{-(pn+2)} dx. \quad (5)$$

If the coefficient of x is nonzero, the integrand grows exponentially at one end and the integral diverges. If it is zero, the exponent $pn+2 \geq 2$ makes the integral finite. The condition is exactly $p(m-an) = 2(a-1)$. For irrational a , two pairs satisfying this condition must agree. Lemma 1.2 now gives the asserted spaces. A rational p in E_a would give $a = (pm+2)/(pn+2) \in \mathbb{Q}$, a contradiction.

The classical density of the fractional parts of an implies that $\{m-an > 0 : m, n \in \mathbb{N}_0\}$ is dense in $(0, \infty)$. More explicitly, for a target $k+t$ with integer $k \geq 0$ and $0 < t < 1$, choose arbitrarily large n with $1 - \{an\}$ close to t , and set $m = \lceil an \rceil + k$. Integer targets follow by approximation. Applying the homeomorphism $s \mapsto 2(a-1)/s$ proves density of E_a . $\square$

Corollary 2.2. *For every irrational $p > 0$, the choice $a = 1 + p/2$ gives $A^p(\Omega_a) = \text{span}\{z\}$ and*

$$\int_{\Omega_a} |z|^p dV = \frac{\pi^3}{8}.$$

Its A^r space is zero for every positive rational r , in particular for $r = 2$.

Proof. In (5), take $m = 1, n = 0$ and $a = 1 + p/2$. The remaining integral is $2\pi^2 \int_{\mathbb{R}} (4+x^2)^{-2} dx = 2\pi^2(\pi/16)$. $\square$

Together with $\mathbb{C}^2$ and a bidisc, which have dimensions zero and infinity, respectively, this proves the realizability assertion in Theorem 1.1. The density of E_a does not mean an individual function is integrable for an interval of exponents: each nonconstant monomial in this example is integrable at at most one exponent. The domain has infinite volume, so no finite-measure nesting principle for L^p spaces applies.

3 Geometry of the zero Bergman space

The ordinary Bergman space of Ω_a vanishes, but its logarithmic image has no affine line. This distinction is relevant to geometric questions about complete Reinhardt domains with trivial Bergman space; without pseudoconvexity, the actual logarithmic image must not be replaced by its convex hull in the affine-line criterion.

Proposition 3.1. *For irrational $a > 1$, the logarithmic image of Ω_a is*

$$L_a = \{(x, y) : y < -ax - \log(4+x^2)\}.$$

It contains no real affine line, while

$$\text{conv } L_a = \{(x, y) : ax + y < -\log 4\}.$$

Every bounded holomorphic function on Ω_a is constant. Every nonconstant holomorphic map $\mathbb{C} \rightarrow \Omega_a$ has image in one of the coordinate axes.

Proof. The formula for L_a follows from the definition. A vertical line cannot lie in it. Containment of a nonvertical line $y = bx + c$ would require $(a + b)x + c < -\log(4 + x^2)$ for all real x . If $a + b$ is nonzero, this fails at one end; if it is zero, it fails at both ends. Thus there is no affine line.

Put $h(x) = -\log(4 + x^2)$ and apply the linear change $(x, y) \mapsto (x, ax + y)$. The transformed set lies below $h(0) = -\log 4$. For a fixed $x \neq 0$, take R of the same sign with $|R| > |x|$. The chord joining points just below $(0, h(0))$ and $(R, h(R))$ has height arbitrarily close to $(1 - x/R)h(0) + (x/R)h(R)$ at x . This tends to $h(0)$ as $|R| \rightarrow \infty$. The convex hull is downward closed, so it contains every point strictly below that height; the case $x = 0$ already lies in the set. This proves the claimed open half-plane, not just its closure. In particular L_a is not convex (also $h''(x) > 0$ for $|x| > 2$), so the domain is not pseudoconvex.

If a bounded holomorphic function has a nonzero Taylor coefficient c_{mn} , torus integration shows that $|z^m w^n|$ is bounded on Ω_a . Taking $|w|$ up to its fiber radius reduces this to boundedness of $e^{(m-an)x}(4 + x^2)^{-n}$ for all real x . It forces $m = an$, and irrationality gives $m = n = 0$.

Finally let $(f, g) : \mathbb{C} \rightarrow \Omega_a$ be holomorphic, with neither component identically zero. The subharmonic function $a \log |f| + \log |g|$ is bounded above by $-\log 4$, and therefore is constant by the subharmonic Liouville theorem. It is finite somewhere, hence finite everywhere; neither f nor g has zeros. Write $f = e^F, g = e^G$ with entire F, G . The positive constant $|f|^a |g|$ is at most $(4 + (\operatorname{Re} F)^2)^{-1}$. Thus $\operatorname{Re} F$ is bounded, so F is constant. The same identity bounds $\operatorname{Re} G$, making G constant as well. A nonconstant map must therefore have one component identically zero. $\square$

4 Prescribed dimensions at least two in three variables

The dimension-two restriction does not persist in dimension three.

Theorem 4.1. *Fix an irrational $p > 0$ and an integer $k \geq 2$. Set $d = k - 1$, $a = (pd + 4)/2$, and, for $u = (z_1, z_2) \in \mathbb{C}^2$, define*

$$D_{p,k} = (\{(0, 0)\} \times \mathbb{C}) \cup \{(u, w) : u \neq 0, |w| < F_a(\|u\|)\}.$$

Then $D_{p,k} \subset \mathbb{C}^3$ is an unbounded complete Reinhardt domain with smooth boundary, and

$$A^p(D_{p,k}) = \operatorname{span}\{z_1^i z_2^{d-i} : 0 \leq i \leq d\}.$$

In particular its dimension is exactly k .

Proof. The monotonicity of F_a proves coordinatewise completeness, and its limit at zero proves openness on the added axis. Contractions give connectedness. A finite boundary point has $u \neq 0, w \neq 0$, where the same defining-function argument as before proves smoothness. For a monomial $z_1^i z_2^j w^\ell$, integration over the w -disc and polar coordinates $u = r\omega$ in $\mathbb{C}^2$ give a positive finite angular factor times

$$\int_{\mathbb{R}} e^{[p(i+j)+4-a(p\ell+2)]x} (4 + x^2)^{-(p\ell+2)} dx.$$

The angular factor is the integral of $|\omega_1|^{pi} |\omega_2|^{pj}$ over the unit sphere, multiplied by $2\pi/(p\ell + 2)$; it is finite and nonzero. Thus integrability holds precisely when

$$i + j = d + \left(\frac{pd}{2} + 2\right) \ell.$$

Since $d \geq 1$ and p is irrational, this equality of nonnegative integers forces $\ell = 0$ and $i + j = d$. The residual integral has power 2 and converges. Lemma 1.2 finishes the proof. $\square$

5 Scope and reproducibility

The all-dimensional rational-exponent restriction is an elementary bounded-multiplier argument. The finer statements are the full list of possible dimensions in $\mathbb{C}^2$, the dense exceptional spectrum on one fixed domain, and the explicit $\mathbb{C}^3$ realizations for $k \geq 2$. These results do not classify all complete Reinhardt domains with trivial A^2 , nor do they assert that finite positive dimension occurs on pseudoconvex domains.

Coordinatewise completeness is essential. For example, the union displayed in [5, Section 5] contains (T, T) for all large T , but not its coordinate contraction $(T, T/2)$. For $T > 4e$, the latter point lies in neither the bounded polydisc, either thin axial region, nor the thin diagonal band. That displayed union is Reinhardt but not complete in the sense of (1). This observation concerns that construction, not the other results in the cited work.

The accompanying program checks the determinant identities, the rational multiplier exponents, the fiber derivative bound and finite critical-index sets using exact rational arithmetic and $\mathbb{Q}(\sqrt{3})$. Its normal and optimized runs agree. A separate 80-digit quadrature checks the norm in Corollary 2.2; it is a numerical regression, not a certified enclosure. None of the general proofs or density conclusions is inferred from a finite sweep.

AI tools assisted with derivation, literature comparison, code and manuscript preparation. This preprint is self-audited and unrefereed; the computations are not independent human review or formal verification. Classical monomial methods, convex geometry and irrational-rotation density are used with no absolute-priority claim.

References

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