

Convex Near-Extremizers for the Multidimensional Hermite–Hadamard Inequality

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Abstract

Larson proved that the boundary-mean Hermite–Hadamard inequality on bounded convex domains in $\mathbb{R}^n$ has sharp constant n for nonnegative subharmonic functions. We give explicit nonnegative convex functions on collapsing simplices whose ratios tend to the same constant. Consequently, restricting the test functions to the convex class does not improve the universal constant. The construction uses the positive part of an affine function, with its localization scale larger than the collapsing thickness. We compute its exact ratio and a quantitative error bound. The two volume-power normalized sharp-constant problems are not resolved by this construction.

1 The sharp constant for convex functions

Let $\Omega \subset \mathbb{R}^n$ be a bounded nonempty open convex set, and let $f \in C(\overline{\Omega})$ be nonnegative, convex and not identically zero. Write $|\Omega|$ for volume and $|\partial\Omega|$ for Euclidean $(n-1)$ -dimensional Hausdorff measure. Define

$$\mathcal{R}(\Omega, f) = \frac{|\partial\Omega|}{|\Omega|} \frac{\int_{\Omega} f(x) dx}{\int_{\partial\Omega} f(x) d\sigma(x)}, \quad C_n^{\text{cvx}} = \sup_{\Omega, f} \mathcal{R}(\Omega, f). \quad (1)$$

The denominator is positive: if a nonnegative continuous convex function vanishes on the boundary of a convex body, convexity forces it to vanish everywhere.

Larson’s theorem [3, Theorem 1.1] gives $C_n^{\text{cvx}} \leq n$, since convex functions are subharmonic. Its sharpness statement concerns the larger subharmonic class. We supply convex near-extremizers.

Theorem 1. *For every integer $n \geq 2$, one has $C_n^{\text{cvx}} = n$. The same supremum is obtained if the test functions are required to be smooth, strictly positive and strictly convex on a neighborhood of the domain closure.*

Steinerberger [1] initiated this higher-dimensional problem. Beck et al. [2, Theorem 2 and Section 4] obtained a lower bound $n - 1$ using the positive part of an affine function. Our contribution is the two-scale simplex family below and its exact ratio, not the hinge-function idea or Larson’s upper bound. The theorem answers the boundary-mean convex part of AIM problem 6.1 [4]. That source asks for four constants; the present note does not determine either volume-power normalized constant.

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2 An exact simplex calculation

Put $k = n - 1$ and $s = x_1 + \cdots + x_k$. For $h > 0$ and $0 < d \leq 1$ set

$$K_h = \{(x, y) \in \mathbb{R}^k \times \mathbb{R} : x_i \geq 0, y \geq 0, s + y/h \leq 1\}, \quad f_d(x, y) = (d - s)_+. \quad (2)$$

Here $t_+ = \max(t, 0)$. The domain is the interior of K_h , the simplex with vertices $0, e_1, \dots, e_k, he_n$. The function is globally continuous, convex and nonnegative.

Proposition 2. *Let $A = 1 + \sqrt{1 + (n - 1)h^2}$. Then*

$$\mathcal{R}(\text{int } K_h, f_d) = \frac{n[A + (n - 1)h] \left(1 - \frac{n-1}{n+1}d\right)}{A + (n - 1)\frac{h}{d}\{n - (n - 2)d\}}. \quad (3)$$

Proof. Write $V = |K_h|$, $S = |\partial K_h|$, and $q = \sqrt{1 + kh^2}$. The bottom face has area $1/k!$, the sloping face has area $q/k!$, and each of the k vertical coordinate faces has area $h/k!$. Therefore

$$V = \frac{h}{n!}, \quad S = \frac{A + kh}{(n - 1)!}. \quad (4)$$

For $m \geq 1$, integration in $t = z_1 + \cdots + z_m$ on the positive simplex, with slice density $t^{m-1}/(m-1)!$, gives

$$\int_{\substack{z_i \geq 0 \\ \sum z_i \leq d}} (d - \sum z_i) dz = \frac{d^{m+1}}{(m + 1)!}, \quad (5)$$

$$\int_{\substack{z_i \geq 0 \\ \sum z_i \leq d}} (\sum z_i)(d - \sum z_i) dz = \frac{m d^{m+2}}{(m + 2)!}. \quad (6)$$

For $m = 0$, the integrals mean evaluation at the zero-dimensional simplex and equal d and 0 , respectively; the formulas still hold.

The vertical fiber over x has length $h(1 - s)$. Thus

$$I := \int_{K_h} f_d = \frac{hd^n}{n!} \left(1 - \frac{n-1}{n+1}d\right). \quad (7)$$

The bottom and sloping faces contribute $Ad^n/n!$ to the boundary integral, since the sloping graph has constant area Jacobian q . On a vertical coordinate face there are $n - 2$ horizontal coordinates, so its contribution is

$$\frac{hd^{n-1}}{(n - 1)!} \left(1 - \frac{n-2}{n}d\right).$$

This includes $n = 2$, when the face is a segment and the contribution is hd . Adding all faces yields

$$J := \int_{\partial K_h} f_d d\sigma = \frac{d^{n-1}}{n!} [dA + (n - 1)h\{n - (n - 2)d\}]. \quad (8)$$

The support cut $s = d$ is not a face of ∂K_h and adds no boundary term. Substitution of (4)–(8) into $SI/(VJ)$ proves (3). $\square$

Proof of Theorem 1. For each fixed n , choose $h = d^2$ and let $d \downarrow 0$. Then $A \rightarrow 2$ and $h/d \rightarrow 0$, so (3) gives

$$\lim_{d \downarrow 0} \mathcal{R}(\text{int } K_{d^2}, f_d) = n. \quad (9)$$

Every parameter pair gives a full-dimensional convex domain and positive integrals. Consequently every constant smaller than n fails. Larson's upper bound proves $C_n^{\text{cvx}} = n$.

For the additional test-function restrictions, fix h, d first and replace f_d by

$$f_{d,\varepsilon}(x, y) = \frac{d - s + \sqrt{(d - s)^2 + \varepsilon^2}}{2} + \varepsilon|(x, y)|^2, \quad \varepsilon > 0.$$

This function is smooth, positive and strictly convex on $\mathbb{R}^n$. It converges uniformly to f_d on the fixed compact simplex, so both integrals and their ratio converge. Choosing ε after h, d preserves any strict lower bound below the limiting value n . $\square$

3 Quantitative form and scope

Let $R_n(d) = \mathcal{R}(\text{int } K_{d^2}, f_d)$. Since $A \geq 2$,

$$\frac{R_n(d)}{n} \geq \frac{1 - \frac{n-1}{n+1}d}{1 + \frac{n(n-1)}{2}d} \geq 1 - (n-1) \left(\frac{1}{n+1} + \frac{n}{2} \right) d.$$

Together with Larson's strict bound for each fixed domain, this gives

$$0 < n - R_n(d) \leq n(n-1) \left(\frac{1}{n+1} + \frac{n}{2} \right) d \quad (0 < d \leq 1). \quad (10)$$

The strict upper inequality can also be checked directly in (3). The two scales matter: fixing $d = 1$ and letting $h \rightarrow 0$ yields only $2n/(n+1)$, while fixing $h > 0$ and sending $d \rightarrow 0$ makes the ratio tend to zero. The required regime is $h = o(d)$.

In the plane, the construction is the triangle $\text{conv}\{(0,0), (1,0), (0,h)\}$ with $f_d = (d - x)_+$. Direct edge and fiber integration gives

$$V = \frac{h}{2}, \quad S = 1 + h + \sqrt{1 + h^2}, \quad I = \frac{hd^2}{2}(1 - d/3), \quad J = \frac{d^2}{2}(1 + \sqrt{1 + h^2}) + hd.$$

The choice $h = d^2$ gives a ratio tending to 2, rather than the fixed-affine limit $4/3$.

For the separate volume-power normalization the ratio is

$$\frac{I}{V^{1/n}J} = \frac{V^{(n-1)/n}}{S} \frac{SI}{VJ}.$$

It tends to zero in our family. Thus neither volume-power sharp constant follows from Theorem 1. Recent work on torsion-gradient optimizers under volume or perimeter constraints [5, 6, 7] concerns a different optimization problem.

The frozen dataset record AIM-ANALYSIS-0032 in `ulamai/UnsolvedMath` v1.6.0 [8] therefore remains partial, even though its boundary-mean convex subproblem is resolved by the theorem. No absolute-priority claim is made.

The accompanying standard-library Python program performs 4,072 exact arithmetic checks, including an independent triangulation-based integration of the clipped simplices and Gram-determinant face areas. Normal and optimized runs agree. These finite regressions supplement the all-dimensional argument; they are not formal verification. AI-assisted tools were used for derivation, literature retrieval, exact computation and manuscript preparation. This note is unrefereed; no independent review is asserted.

References

- [1] S. Steinerberger, The Hermite–Hadamard inequality in higher dimensions, *J. Geom. Anal.* **30** (2020), 466–483. arXiv:1808.07794.
- [2] T. Beck, B. Brandolini, K. Burdzy, A. Henrot, J. J. Langford, S. Larson, R. Smits and S. Steinerberger, Improved bounds for Hermite–Hadamard inequalities in higher dimensions, *J. Geom. Anal.* **31** (2021), 801–816. arXiv:1907.06122v1.
- [3] S. Larson, A sharp multidimensional Hermite–Hadamard inequality, *Int. Math. Res. Not. IMRN* **2022** (2), 1297–1312. doi:10.1093/imrn/rnaa138; arXiv:2005.01853v2.
- [4] American Institute of Mathematics, *Shape optimization with surface interactions* (2019), Problem 6.1, Hermite–Hadamard inequality in higher dimensions.
<http://aimpl.org/shapesurface/6/>.
- [5] L. Huang, On convex domains maximizing the gradient of the torsion function, arXiv:2504.07340v1 (2025).
- [6] K. Burdzy, I. Ftouhi and P. Mariano, Geometric properties of optimizers for the maximum gradient of the torsion function, arXiv:2512.09400v2 (2025).
- [7] M. van den Berg, D. Bucur and M. Nahon, Shape optimisation of the maximum of the gradient of the torsion function on convex sets, arXiv:2610.08408v1 (2026).
- [8] ulamai, *UnsolvedMath*, version 1.6.0, record AIM-ANALYSIS-0032.
<https://huggingface.co/datasets/ulamai/UnsolvedMath>.