

Uniform Weak Robin Comparisons on Triangles and a Cubic Moment Formula

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Abstract

We give a quantitative comparison for the first Robin eigenvalue among triangles of fixed area. There exists an absolute positive constant ε such that, whenever $0 < |\alpha|\sqrt{A} < \varepsilon$, the eigenvalue difference from the equilateral triangle, divided by α , is at least one twenty-fourth of the perimeter deficit divided by A . The parameter interval is common to all triangle shapes, including degenerating families. The proof combines analytic perturbation near the equilateral triangle, compactness on bounded-perimeter families, and a one-dimensional estimate on thin triangles. We explicitly credit earlier attractive-Robin comparisons and width estimates. Separately, for each fixed domain circumscribed about a ball, we express the cubic Robin coefficient as the variance of squared distance to the center divided by the cube of the inradius. Neither result settles the triangle optimization problem at arbitrary nonzero Robin parameter.

1 A common weak-coupling interval

Let $T \subset \mathbb{R}^2$ be the interior of a nondegenerate triangle of area A . Write $P(T)$ for its perimeter and E_A for an equilateral triangle of the same area. We use the outward-normal convention $\partial_\nu u + \alpha u = 0$, so that

$$\lambda_1(T, \alpha) = \inf_{u \in H^1(T) \setminus \{0\}} \frac{\int_T |\nabla u|^2 + \alpha \int_{\partial T} |u|^2}{\int_T |u|^2}. \quad (1)$$

Boundary integrals are understood as traces. The discrete Bareket conjecture asks whether E_A minimizes this eigenvalue for every $\alpha > 0$ and maximizes it for every $\alpha < 0$; see AIM Conjecture 1.3 [6] and [1, Conjecture 1].

Theorem 1. *There exists an absolute constant $\varepsilon > 0$ such that every nondegenerate triangle T of area A satisfies*

$$\frac{\lambda_1(T, \alpha) - \lambda_1(E_A, \alpha)}{\alpha} \geq \frac{P(T) - P(E_A)}{24A}, \quad 0 < |\alpha|\sqrt{A} < \varepsilon. \quad (2)$$

Consequently E_A is the unique minimizer for positive α and the unique maximizer for negative α in that interval, up to rigid motions. The constant ε is existential; no numerical value is asserted.

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Krejčířík, Lotoreichik and Vu [1] prove local optimality for weak attractive coupling and global comparisons with shape-dependent parameter ranges. Their Theorem 4.2 is uniform on compact shape families. Together with a constant trial function on large-perimeter triangles, it already yields the qualitative negative weak-coupling conclusion of Theorem 1. We do not claim that conclusion independently of their work. Our formulation gives a single two-sided quantitative estimate and a proof of the positive weak-coupling comparison with control of all degenerating shapes.

The slicing principle in our thin-triangle estimate is also known: Bucur and Fragalà [3, Proposition 10] bound positive Robin eigenvalues using the width of a convex domain. We use an elementary one-endpoint version to keep the constants explicit. Laugesen [4] discusses related polygonal questions, and Carboti, Cito and Pallara [5] study existence and qualitative properties of optimal generalized polygons. These existence results do not identify the optimizer in the all-parameter triangle problem.

The distinction between a fixed-shape expansion and a shape-uniform comparison is essential. The leading coefficient at $\alpha = 0$ is $P(T)/A$, but the perimeter deficit vanishes near E_A , and the fixed-area shape space is not compact. We address these two issues separately.

2 Analytic perturbation and the equilateral shape

By the scaling law

$$\lambda_1(tT, \alpha) = t^{-2} \lambda_1(T, t\alpha), \quad (3)$$

we may first impose $A = 1$. Set $E = E_1$, centered at its centroid, and let $\ell = P(E)/3$ be its side length. Let $\mathcal{S}_0$ denote the two-dimensional space of real symmetric trace-zero 2×2 matrices. Put

$$T_M = e^M E, \quad M \in \mathcal{S}_0.$$

Every area-one triangle is represented in this way modulo rigid motions. Indeed, an orientation-preserving affine map from E to the triangle has determinant one; its polar decomposition has a positive-definite symmetric factor e^M with $\text{tr } M = 0$.

If t_i is a unit tangent to the i th edge e_i of E , pullback to E transforms (1) into the form

$$q_{M,\alpha}[u] = \int_E \nabla u^\top e^{-2M} \nabla u + \alpha \sum_{i=1}^3 |e^M t_i| \int_{e_i} |u|^2, \quad u \in H^1(E). \quad (4)$$

The mass form remains $\int_E |u|^2$.

Lemma 2. *For $\alpha \neq 0$, define $F(M, \alpha) = \lambda_1(T_M, \alpha)/\alpha$. It has a jointly real-analytic extension near every $(M, 0)$, with*

$$F(M, 0) = P(T_M). \quad (5)$$

On each compact set of shape parameters, $F(\cdot, \alpha)$ converges to $P(T)$ in C^2 as $\alpha \rightarrow 0$, on one common interval.

Proof. The coefficients in (4) are real analytic, and the trace terms are bounded forms on $H^1(E)$. At $\alpha = 0$ the zero eigenvalue is simple, with constant eigenfunction. For clarity, the analytic branch can be obtained by normalizing $u = 1 + w$ with $\int_E w = 0$ and applying the analytic implicit function theorem to the weak eigenvalue equation. Its derivative in (w, λ) at $(0, 0)$ maps (z, μ) to

$$\phi \mapsto \int_E \nabla z^\top e^{-2M} \nabla \phi - \mu \int_E \phi.$$

Testing with 1 determines μ . Poincaré's inequality and Lax–Milgram then determine the mean-zero z . Thus this derivative is an isomorphism from the mean-zero $H^1(E)$ space times $\mathbb{R}$ onto $H^1(E)^*$.

The resulting branch remains the first eigenvalue for sufficiently small $|\alpha|$, by the isolated simple Neumann eigenvalue and form continuity. Differentiation against the constant eigenfunction gives $\partial_\alpha \lambda_1(T_M, 0) = P(T_M)$, since the area is one. Division by α is therefore an analytic removable singularity. On a compact shape set, uniform ellipticity and Poincaré's inequality give a uniform positive Neumann gap. A finite cover of the local analytic neighborhoods gives a common interval and the stated uniform derivative convergence. The fixed-domain perturbation background is also developed in [2, Proposition 3.1]. $\square$

Lemma 3. *There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that*

$$F(M, \alpha) - F(0, \alpha) \geq \frac{1}{2}(P(T_M) - P(E)) \quad \text{if } \|M\| \leq \delta, \quad |\alpha| < \varepsilon_0. \quad (6)$$

Proof. For a rotation R by 120 degrees, $RE = E$ and $F(RMR^\top, \alpha) = F(M, \alpha)$. Conjugation by R has no fixed vector in $\mathcal{S}_0$ except zero. Hence

$$D_M F(0, \alpha) = 0 \quad (7)$$

throughout a sufficiently small parameter interval.

The perimeter Hessian is positive definite. Specifically, for $H \in \mathcal{S}_0$ write $H^2 = \rho^2 I$ and $q_i = t_i^\top H t_i$. Direct differentiation gives

$$\left. \frac{d^2}{ds^2} |e^{sH} t_i| \right|_{s=0} = 2\rho^2 - q_i^2, \quad \sum_i q_i^2 = \frac{3}{2}\rho^2.$$

Consequently

$$D_M^2 P(T_0)[H, H] = \frac{9}{2}\ell\rho^2 = \frac{9}{4}\ell \operatorname{tr}(H^2). \quad (8)$$

Choose a closed ball $\|M\| \leq \delta$ where the perimeter Hessian is bounded below by $c\|H\|^2$ for some $c > 0$. Lemma 2 permits ε_0 to be chosen so that the Hessian of $F(M, \alpha) - P(T_M)$ has operator norm at most $c/2$ on this ball. Both its gradient and the perimeter gradient vanish at zero. Taylor's integral formula along sM now yields

$$|F(M, \alpha) - F(0, \alpha) - P(T_M) + P(E)| \leq \frac{1}{2}(P(T_M) - P(E)),$$

which proves (6). $\square$

3 Compact shapes and thin triangles

Lemma 4. *For every fixed $K > P(E)$ there exists $\varepsilon_K > 0$ such that every area-one triangle of perimeter at most K satisfies*

$$\frac{\lambda_1(T, \alpha) - \lambda_1(E, \alpha)}{\alpha} \geq \frac{1}{2}(P(T) - P(E)), \quad 0 < |\alpha| < \varepsilon_K. \quad (9)$$

Proof. This shape family is compact modulo rigid motions. For example, E contains a fixed centered disk, so bounded diameter of $e^M E$ bounds the largest singular value of e^M . Its determinant

is one, which also bounds the smallest singular value away from zero. Thus the corresponding matrices M lie in a compact set.

Lemma 3 handles a neighborhood of $M = 0$. On the compact complement of that neighborhood, the triangle isoperimetric inequality makes $P(T) - P(E)$ bounded below by a positive number. Uniform convergence in Lemma 2 then proves (9), after reducing the common interval. Taking the minimum of the two positive thresholds completes the proof. $\square$

Lemma 5. *If T has area one and $0 < \alpha \leq 1$, then*

$$\frac{\lambda_1(T, \alpha)}{\alpha} \geq \frac{P(T)}{12}. \quad (10)$$

Proof. Choose a longest side as a horizontal base of length a , and put $h = 2/a$. The apex projects into the base: otherwise a side joining it to the more distant endpoint would exceed a . Almost every vertical section has length $L \leq h$ and its lower endpoint on the base. For $f \in H^1(0, L)$, the fundamental theorem of calculus and Cauchy-Schwarz give

$$\int_0^L |f(y)|^2 dy \leq 2L|f(0)|^2 + L^2 \int_0^L |f'(y)|^2 dy. \quad (11)$$

Integrate over the base. If $\alpha h \leq 2$, multiplication by $\alpha/(2h)$ bounds the mass by the base boundary term and the vertical derivative energy. The remaining terms in (1) are nonnegative. Therefore

$$\lambda_1(T, \alpha) \geq \frac{\alpha}{2h}, \quad \frac{\lambda_1(T, \alpha)}{\alpha} \geq \frac{a}{4} \geq \frac{P(T)}{12}.$$

Finally, $h \leq a$ and $ah = 2$ imply $h \leq \sqrt{2}$, so $0 < \alpha \leq 1$ suffices for $\alpha h \leq 2$. $\square$

Proof of Theorem 1. Continue with area one and set $K = 24P(E)$. Lemma 4 gives a stronger estimate than (2) when $P(T) \leq K$.

If $P(T) \geq K$ and $0 < \alpha \leq 1$, the constant trial on E gives $F(0, \alpha) \leq P(E)$. Lemma 5 gives

$$F(T, \alpha) - F(E, \alpha) \geq \frac{P(T)}{12} - P(E) \geq \frac{P(T)}{24} \geq \frac{P(T) - P(E)}{24},$$

where $F(T, \alpha)$ denotes $\lambda_1(T, \alpha)/\alpha$. For negative α , the constant trial on T instead gives $F(T, \alpha) \geq P(T)$, because division reverses the inequality. Since $F(E, \alpha) \rightarrow P(E)$, we may require $F(E, \alpha) \leq 2P(E)$ on a small negative interval. On $P(T) \geq K$ this gives a stronger estimate than the desired one.

Choose the minimum of all these positive thresholds. For general A , write $T = \sqrt{A}T_1$ and $\beta = \alpha\sqrt{A}$ in (3). The quotient of eigenvalues scales by $A^{-1/2}$, as does $(P(T) - P(E_A))/A$. This proves (2). The strict perimeter inequality for non-equilateral triangles proves the uniqueness assertion for both signs. $\square$

4 A cubic coefficient on circumsolids

A second calculation gives more precise fixed-domain information. Let $\Omega \subset \mathbb{R}^d$ be a bounded connected Lipschitz domain. Suppose that, for a point I and $r > 0$,

$$(x - I) \cdot \nu(x) = r \quad \text{for almost every } x \in \partial\Omega. \quad (12)$$

A convex polytope circumscribed about a ball of radius r has this property; in particular every triangle does. Write $\langle g \rangle = |\Omega|^{-1} \int_{\Omega} g$.

Proposition 6. *If X is uniformly distributed in Ω and $m = \mathbb{E}|X - I|^2$, then*

$$\lambda_1(\Omega, \alpha) = \frac{d}{r}\alpha - \frac{m}{r^2}\alpha^2 + \frac{\text{Var}(|X - I|^2)}{r^3}\alpha^3 + O_\Omega(\alpha^4). \quad (13)$$

The cubic coefficient is positive. The remainder is for each fixed domain and is not asserted to be uniform on all triangle shapes.

Proof. Normalize the analytic first eigenfunction to have mean one and write

$$u_\alpha = 1 + \alpha v + \alpha^2 w + O(\alpha^3), \quad \langle v \rangle = \langle w \rangle = 0, \quad \lambda_1 = a\alpha + b\alpha^2 + c\alpha^3 + O(\alpha^4).$$

The first corrector satisfies $-\Delta v = a$ and $\partial_\nu v = -1$. The divergence theorem and (12) give $|\partial\Omega|/|\Omega| = d/r$, so

$$a = \frac{d}{r}, \quad v(x) = \frac{m - |x - I|^2}{2r}. \quad (14)$$

This quadratic circumsolid corrector is known from [2, Section 8]; it is not a new ingredient here.

Testing the eigenvalue equation against 1 and using the mean-one normalization gives $b = |\Omega|^{-1} \int_{\partial\Omega} v$ and $c = |\Omega|^{-1} \int_{\partial\Omega} w$. The order-one weak equation tested against v therefore gives

$$b = -\langle |\nabla v|^2 \rangle = -\frac{m}{r^2}.$$

Next, test the order-two equation against v and the order-one equation against w . The resulting identities are

$$\begin{aligned} \int_{\Omega} \nabla w \cdot \nabla v + \int_{\partial\Omega} v^2 &= a \int_{\Omega} v^2, \\ \int_{\Omega} \nabla v \cdot \nabla w + \int_{\partial\Omega} w &= 0. \end{aligned}$$

Eliminating the gradient term yields

$$c = \frac{1}{|\Omega|} \left(\int_{\partial\Omega} v^2 - \frac{d}{r} \int_{\Omega} v^2 \right). \quad (15)$$

The divergence of $(x - I)v^2$ and $\nabla v = -(x - I)/r$ turn (15) into

$$c = -\frac{2}{r^2} \langle v|x - I|^2 \rangle = \frac{\langle |x - I|^4 \rangle - m^2}{r^3}.$$

The variance is strictly positive because squared distance is not constant on any nonempty open domain. Analyticity gives the fixed-domain remainder in (13). $\square$

5 Triangle moments and scope

For a triangle, put $Q(T) = m/r^2$. If G is its centroid and a, b, c are its side lengths, the parallel-axis identity gives

$$m = \frac{a^2 + b^2 + c^2}{36} + |G - I|^2.$$

Since $r = 2A/P(T)$ and $P(E_A)^4 = 432A^2$, Cauchy-Schwarz yields

$$Q(T) \geq \frac{P(T)^2(a^2 + b^2 + c^2)}{144A^2} \geq \left(\frac{P(T)}{P(E_A)} \right)^4 \geq 1. \quad (16)$$

Equality requires the equilateral shape. The second-order term is therefore unfavorable for a direct positive-parameter comparison; the uniform theorem cannot be obtained just by keeping two terms.

For exact moment evaluation, let x, y, z be the positive tangency lengths at the vertices, and put $s = x + y + z$. Then

$$r^2 = \frac{xyz}{s}, \quad m = r^2 + \frac{s^2 - 3(xy + yz + zx)}{6}.$$

The vertex Gram matrix relative to I has diagonal entries $r^2 + x_i^2$ and off-diagonal entries $r^2 - x_i x_j$. Integrating its barycentric quadratic form and its square computes both m and the variance in Proposition 6. For an equilateral triangle, $m = r^2$ and $\text{Var}(|X - I|^2) = 3r^4/5$, whence

$$\lambda_1(E_A, \alpha) = \frac{2}{r}\alpha - \alpha^2 + \frac{3r}{5}\alpha^3 + O(\alpha^4).$$

The accompanying standard-library Python program checks the moment and boundary identities with exact rational arithmetic on all $1 \leq x, y, z \leq 12$, together with ball-moment and perimeter-Hessian checks. All 6,933 checks agree in normal and optimized execution. They supplement the analytic proof and are not formal verification.

Theorem 1 gives no estimate for the missing moderate couplings or a common interval extending to infinity. Accordingly, the frozen record AIM-ANALYSIS-0019 remains partial. It is recorded in `ulamai/UnsolvedMath` v1.6.0 [7]. The manuscript supplies complete proofs of the stated weak-coupling and moment results, not a complete solution of that source record. No absolute-priority claim is made. AI-assisted tools were used for derivation, literature retrieval, exact calculation and manuscript preparation. This preprint is unrefereed; no independent review is asserted.

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