

# A Four-Point Obstruction to Completeness for Three Atomic Inner Functions

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## Abstract

For  $\xi \in \mathbb{T}$  and  $\alpha > 0$ , let  $u_{\xi,\alpha}(z) = \exp[-\alpha(\xi + z)/(\xi - z)]$ . We give an explicit four-point functional annihilating every individual nonnegative power of  $u_{1,3\pi}$ ,  $u_{-1,3\pi}$  and  $u_{i,3\pi}$ , but not  $z^2$ . Thus their linear span is not weak-star dense in  $H^\infty$ , not dense in any  $H^p$ ,  $0 < p < \infty$ , and not dense for locally uniform convergence. The same certificate works with the fourth atom at  $-i$  and with independent positive integer multiples of the masses. A real smooth Poisson-kernel annihilator also obstructs bilateral weak-star completeness in  $L^\infty(\mathbb{T})$ . Four is the smallest possible support size for an evaluation annihilator associated with three distinct atomic singularities. Conformal transport supplies bad positive masses at any prescribed triple of distinct boundary locations. These results disprove universal completeness, but do not classify the complete parameter configurations.

## 1 The approximation question

Write  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ ,  $\mathbb{T} = \partial\mathbb{D}$ , and

$$C_\xi(z) = \frac{\xi + z}{\xi - z}, \quad u_{\xi,\alpha}(z) = e^{-\alpha C_\xi(z)} \quad (\xi \in \mathbb{T}, \alpha > 0). \quad (1)$$

These are atomic singular inner functions. For a finite list  $\mathcal{U} = (u_1, \dots, u_r)$ , set

$$E(\mathcal{U}) = \text{span}_{\mathbb{C}}\{u_j^m : 1 \leq j \leq r, m = 0, 1, 2, \dots\}. \quad (2)$$

Mixed products of different  $u_j$  are not included.

The two-atom theory has a sharp answer. For atoms at  $1, -1$  with masses  $\alpha, \beta$ , Hedenmalm and Montes-Rodriguez obtain completeness precisely when  $\alpha\beta \leq \pi^2$ : for bilateral powers in weak-star  $L^\infty$  in [2], and for nonnegative powers in weak-star  $H^\infty$  in [3]. Miheisi gives a more general two-inner-function criterion based on univalence on complementary arcs [4]. Seco's 2014 conference abstract already considers adding a third shifted atomic function to a supercritical pair [5].

Problem 3.2 in the AIM list [1], attributed to Alfonso Montes Rodriguez, asks an approximation question involving three such power families. The archived statement does not specify a target, topology, function space, or coefficient class. We therefore address the explicit universal density assertion for (2), rather than silently assigning a unique meaning to that prompt. Our construction gives noncomplete configurations, including one for every triple of boundary locations after choosing suitable masses. A criterion for all prescribed masses remains outside the result.

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## 2 An explicit square certificate

Put

$$t = 1 - \frac{1}{\sqrt{2}}, \quad \rho = \sqrt{2} - 1, \quad z_{\varepsilon, \delta} = t(\varepsilon + i\delta) \quad (\varepsilon, \delta \in \{-1, 1\}). \quad (3)$$

The four points are distinct, and  $|z_{\varepsilon, \delta}| = \rho < 1$ . Define

$$L(f) = \sum_{\varepsilon, \delta = \pm 1} \varepsilon \delta f(z_{\varepsilon, \delta}). \quad (4)$$

**Theorem 1.** *Let  $\mathcal{U}$  consist of the three functions  $u_{1,3\pi}, u_{-1,3\pi}, u_{i,3\pi}$ , or of these together with  $u_{-i,3\pi}$ . Then*

$$L(g) = 0 \quad (g \in E(\mathcal{U})), \quad L(z^2) = 8it^2 = 4i\rho^2 \neq 0. \quad (5)$$

*The conclusion remains true if each mass  $3\pi$  is independently replaced by  $3\pi N_j$  with  $N_j \in \mathbb{N}$ . In particular, every such  $g$  satisfies*

$$\max_{\varepsilon, \delta = \pm 1} |z_{\varepsilon, \delta}^2 - g(z_{\varepsilon, \delta})| \geq 3 - 2\sqrt{2}. \quad (6)$$

*Proof.* The identities  $1 + 2t^2 = 4t$  and  $1 - 2t^2 = 2\sqrt{2}t$  give

$$C_1(z_{1,\delta}) = \sqrt{2} + i\delta, \quad C_1(z_{-1,\delta}) = \frac{\sqrt{2} + i\delta}{3}. \quad (7)$$

Indeed, the real and imaginary parts of  $C_1(z_{\varepsilon, \delta})$  are respectively  $(1 - 2t^2)/(1 - 2\varepsilon t + 2t^2)$  and  $2\delta t/(1 - 2\varepsilon t + 2t^2)$ ; their denominators are  $2t$  and  $6t$ . Multiplication by  $3\pi$  in (7) makes the imaginary parts odd integer multiples of  $\pi$ . Set

$$a = -e^{-3\sqrt{2}\pi}, \quad b = -e^{-\sqrt{2}\pi}. \quad (8)$$

Using  $C_{-1}(z) = C_1(-z)$ ,  $C_i(z) = C_1(-iz)$  and  $C_{-i}(z) = C_1(iz)$  gives Table 1.

|                    | $z_{1,1}$ | $z_{1,-1}$ | $z_{-1,1}$ | $z_{-1,-1}$ |
|--------------------|-----------|------------|------------|-------------|
| $u_{1,3\pi}$       | $a$       | $a$        | $b$        | $b$         |
| $u_{-1,3\pi}$      | $b$       | $b$        | $a$        | $a$         |
| $u_{i,3\pi}$       | $a$       | $b$        | $a$        | $b$         |
| $u_{-i,3\pi}$      | $b$       | $a$        | $b$        | $a$         |
| Coefficient in $L$ | 1         | -1         | -1         | 1           |

Table 1: Exact values at the four support points.

Every row has zero signed sum after applying any function to its entries. In particular,  $L(u_j^m) = 0$  for every  $m \geq 0$ . Since  $u_{\varepsilon, 3\pi N} = u_{\varepsilon, 3\pi}^N$ , integer multiples of individual masses preserve the cancellation. On the other hand,  $z_{\varepsilon, \delta}^2 = 2i\varepsilon\delta t^2$ , so its signed sum is  $8it^2$ . Finally  $|L(f)| \leq 4 \max |f(z_{\varepsilon, \delta})|$  and  $2t^2 = 3 - 2\sqrt{2}$  imply (6).  $\square$

*Remark 2.* The functional annihilates any sum  $\sum_j F_j(u_j)$  whenever the values of each  $F_j$  at the finitely many relevant arguments are defined. It does not annihilate the algebra generated by the atoms: for the standard masses,  $L(u_{1,3\pi}u_{i,3\pi}) = (a - b)^2 \neq 0$ .

### 3 Topological and quantitative consequences

We use normalized arc measure  $dm$  on  $\mathbb{T}$ . For  $w \in \mathbb{D}$  put

$$P_w(\zeta) = \frac{1 - |w|^2}{|\zeta - w|^2}, \quad \zeta \in \mathbb{T}. \quad (9)$$

Every bounded analytic  $f$  satisfies  $f(w) = \int_{\mathbb{T}} f(\zeta) P_w(\zeta) dm(\zeta)$ , with boundary values understood almost everywhere.

**Corollary 3.** *For every configuration in Theorem 1,  $z^2$  lies outside the closure of  $E(\mathcal{U})$  in the compact-open topology, in  $H^p$  for each  $0 < p < \infty$ , and in the weak-star topology of  $H^\infty$ . It also lies outside the  $H^\infty$  norm closure.*

*Proof.* Disk evaluations are continuous for locally uniform convergence. For  $0 < p < \infty$ , their continuity on  $H^p$  follows by applying the subharmonic mean inequality to  $|f|^p$  on a circle of radius between  $|w|$  and 1. This proof also applies when  $p < 1$ . Formula (9) represents each evaluation by an  $L^1$  test function, proving weak-star continuity on  $H^\infty$ . Thus  $L$  is continuous in each stated topology, and its proper kernel contains the respective closure. The norm conclusion also follows directly from (6).  $\square$

**Theorem 4.** *For every configuration in Theorem 1, the boundary linear span  $\text{span}\{u_j^m : m \in \mathbb{Z}, 1 \leq j \leq r\}$  is not weak-star dense in  $L^\infty(\mathbb{T})$ . A nonzero real smooth annihilator is*

$$h(\zeta) = \sum_{\varepsilon, \delta = \pm 1} \varepsilon \delta P_{z_{\varepsilon, \delta}}(\zeta). \quad (10)$$

*Proof.* The function  $h$  is real and smooth, and integration against  $h$  equals  $L$  on bounded analytic functions. Therefore  $\int_{\mathbb{T}} u_j^m h dm = 0$  for  $m \geq 0$ . Since  $u_j$  is inner,  $u_j^{-m} = \overline{u_j^m}$  almost everywhere on  $\mathbb{T}$ ; reality of  $h$  gives the same equality for  $m < 0$ . Finally  $\int_{\mathbb{T}} \zeta^2 h(\zeta) dm(\zeta) = 8it^2 \neq 0$ , so this  $L^1$  annihilator is nonzero. Negative powers are used only as boundary functions, not as bounded analytic functions in the disk.  $\square$

There is also a stronger lower bound in the Hilbert-space norm.

**Corollary 5.** *With  $\rho = \sqrt{2} - 1$ , every configuration of Theorem 1 satisfies*

$$\text{dist}_{H^2}(z^2, \overline{E(\mathcal{U})}^{H^2}) \geq \sqrt{1 - \rho^8}. \quad (11)$$

*Proof.* The four support points are  $\rho$  times the fourth roots of  $-1$ , with the alternating signs in (4). Hence

$$L(z^n) = \begin{cases} 4i(-1)^q \rho^{4q+2}, & n = 4q + 2, \ q \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For the  $H^2$  inner product linear in the first entry, the representer is

$$k(z) = \frac{-4i\rho^2 z^2}{1 + \rho^4 z^4}, \quad \|k\|_2^2 = \frac{16\rho^4}{1 - \rho^8}.$$

For  $g$  in the closure in question, Cauchy-Schwarz gives  $\|z^2 - g\|_2 \geq |L(z^2)|/\|k\|_2$ , which is (11). No equality claim is made.  $\square$

## 4 Four support points are necessary

An elementary reconstruction fact explains why a two-point obstruction cannot prove Theorem 1. We include its proof to distinguish point separation from linear completeness.

**Lemma 6.** *For three distinct  $\xi_1, \xi_2, \xi_3 \in \mathbb{T}$  and arbitrary  $\alpha_1, \alpha_2, \alpha_3 > 0$ , the map*

$$z \longmapsto (|u_{\xi_1, \alpha_1}(z)|, |u_{\xi_2, \alpha_2}(z)|, |u_{\xi_3, \alpha_3}(z)|)$$

*is injective on  $\mathbb{D}$ .*

*Proof.* Write  $\xi_j = e^{i\theta_j}$  and  $p_j = -\alpha_j^{-1} \log |u_{\xi_j, \alpha_j}(z)| > 0$ . Directly from (1),

$$\begin{aligned} \frac{1}{p_j} &= A - X \cos \theta_j - Y \sin \theta_j, \\ A &= \frac{1 + |z|^2}{1 - |z|^2}, \quad X = \frac{2 \operatorname{Re} z}{1 - |z|^2}, \quad Y = \frac{2 \operatorname{Im} z}{1 - |z|^2}. \end{aligned}$$

The three rows  $(1, -\cos \theta_j, -\sin \theta_j)$  are linearly independent: a nonzero affine linear function cannot vanish at three distinct points of a circle. Thus  $A, X, Y$ , and hence  $z = (X + iY)/(A + 1)$ , are determined.  $\square$

**Theorem 7.** *For any three distinct atomic singularities with positive masses, a nonzero functional that is a linear combination of evaluations at distinct disk points and annihilates all three nonnegative-power families must have at least four support points. This bound is attained by Theorem 1.*

*Proof.* Suppose  $\Lambda(f) = \sum_{\ell=1}^s c_\ell f(w_\ell)$  is such a functional with  $s \leq 3$ . Discard zero coefficients. For each fixed  $j$ , partition its support by the values of  $u_j(w_\ell)$ , say into fibers with distinct values  $v_1, \dots, v_q$ . If  $b_a$  is the sum of the coefficients in the  $a$ th fiber, then

$$\sum_{a=1}^q b_a v_a^m = 0 \quad (0 \leq m < q).$$

The Vandermonde matrix is invertible, so every  $b_a$  is zero. A singleton fiber is impossible because its coefficient is nonzero. Every partition of two or three points without a singleton has just one part; one support point is already impossible. Thus every  $u_j$  is constant on the entire support. Lemma 6 contradicts distinctness of the support points. This argument concerns evaluations, not derivative evaluations or arbitrary dual functionals.  $\square$

## 5 Prescribed locations and limits of the conclusion

**Theorem 8.** *For any three distinct points  $\xi_1, \xi_2, \xi_3 \in \mathbb{T}$ , there are positive masses  $\alpha_1, \alpha_2, \alpha_3$  for which the pure-power span fails all the density conclusions in Corollary 3, and its bilateral boundary span fails weak-star  $L^\infty$  completeness. A four-point evaluation annihilator witnesses this failure.*

*Proof.* After relabeling the points to match cyclic order, choose a disk automorphism  $\psi$  sending them to  $1, -1, i$ . Write  $\eta_j = \psi(\xi_j)$  and  $d_j = |\psi'(\xi_j)| > 0$ . The Poisson-kernel transformation formula gives

$$\operatorname{Re} C_{\xi_j}(z) = d_j \operatorname{Re} C_{\eta_j}(\psi(z)).$$

The difference of these analytic functions therefore is a constant  $ib_j$  with  $b_j \in \mathbb{R}$ . On setting  $\alpha_j = 3\pi/d_j$ , we get

$$u_{\xi_j, \alpha_j}(z) = e^{-i\alpha_j b_j} u_{\eta_j, 3\pi}(\psi(z)). \quad (12)$$

Pull the four points in (3) back by  $\psi^{-1}$  and keep their real coefficients  $\varepsilon\delta$ . The unimodular constants in (12) do not alter the fiber equalities. The resulting functional annihilates the three power families, but takes the value  $8it^2$  on the bounded analytic function  $\psi(z)^2$ . Continuity of evaluations proves the analytic assertions. The same signed sum of Poisson kernels at the pulled-back points proves the bilateral assertion.  $\square$

Positive integer multiples of each transported mass also work. For four locations the analogous argument applies to automorphic images of  $\{1, -1, i, -i\}$ , not to arbitrary quadruples. In particular, the statements above are existence results for bad masses, not claims that all masses at a given triple fail. They neither replace the sharp two-function theorems nor supply a necessary and sufficient three-function criterion.

The accompanying standard-library Python checker performs exact arithmetic in  $\mathbb{Q}(\sqrt{2}, i)$  and uses symbolic exponential signatures, not floating-point equality tests. It checks the square table, signed fibers, the first 100 representer coefficients, and nontrivial automorphism identities. Its 1,746 checks give identical reports in normal and optimized modes. The proofs of all-power cancellation and topological consequences are the arguments above, not inferences from a finite power cutoff.

This unrefereed note was developed with AI-assisted reasoning, literature search and verification-code preparation. It makes no absolute priority claim and does not claim proof-assistant certification. The earlier three-function question and two-function results are credited above.

## References

- [1] American Institute of Mathematics, *Riemann-Hilbert problems, Toeplitz matrices, and applications*, Section 3, Problem 3.2, attributed to Alfonso Montes Rodriguez. [Archived problem list](#), 28 August 2024.
- [2] H. Hedenmalm and A. Montes-Rodriguez, Heisenberg uniqueness pairs and the Klein-Gordon equation, *Ann. of Math.* **173** (2011), 1507–1527. [Publisher article](#).
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