The Spectrum of the Hilbert Matrix on Power-Weighted ℓ^2 Spaces

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August 29, 2026

Abstract

For $\alpha \in \mathbb{R}$, let ℓ^2_α be the sequence space with squared norm $\sum_{n\geq 0} |x_n|^2(n+1)^\alpha$. We determine the spectrum of the classical Hilbert matrix $H = (m+n+1)^{-1}$ on these spaces, answering a problem from the 2024 AIM workshop on Riemann–Hilbert problems and Toeplitz matrices. The matrix is bounded exactly when $|\alpha| < 1$. In that range its spectrum is the closed lens bounded by

$$\left\{ \frac{\pi}{\cos(\pi|\alpha|/2 + i\pi t)} : t \in \mathbb{R} \right\} \cup \{0\}.$$

The boundary is the Fredholm essential and continuous spectrum. The lens interior is simple point spectrum when $\alpha < 0$ and residual spectrum of defect one when $\alpha > 0$; the interior Fredholm index is $-\operatorname{sgn} \alpha$. The proof compares the diagonally conjugated matrix, modulo a Hilbert–Schmidt operator, with a half-line Wiener–Hopf operator whose symbol is $\pi/\cos(\pi\alpha/2 - i\pi t)$. Hill's complete classification of the latent eigenvectors of the Hilbert matrix then fixes the point and residual parts. This manuscript is unrefereed and makes no absolute priority claim.

1 Statement of the result

For $\alpha \in \mathbb{R}$, put

$$\ell^2_\alpha = \left\{ x = (x_n)_{n\geq 0} : \|x\|^2_\alpha = \sum_{n=0}^{\infty} |x_n|^2(n+1)^\alpha < \infty \right\}. \tag{1}$$

On finitely supported sequences consider

$$(H_\alpha x)_m = \sum_{n=0}^{\infty} \frac{x_n}{m+n+1}. \tag{2}$$

The subscript records the ambient space; the entries of the matrix do not depend on α. This is the bounded-operator interpretation of the AIM problem [2]. For $|\alpha| \geq 1$ an unbounded realization would require a separately specified domain and is not considered here.

For $0 < a < 1$, set $c_a = \pi a/2$ and define

$$\mathcal{C}_a = \left\{ \frac{\pi}{\cos(c_a + i\pi t)} : t \in \mathbb{R} \right\} \cup \{0\}. \tag{3}$$

This is a simple closed curve, symmetric about the real axis, with a corner at zero. Let Ω_a be its bounded interior and $\mathcal{L}_a = \mathcal{C}_a \cup \Omega_a$. Equivalently,

$$\Omega_a = \left\{ \frac{\pi}{\sin(\pi\mu)} : \frac{1-a}{2} < \Re\mu < \frac{1+a}{2} \right\}. \tag{4}$$

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Unrefereed preprint.

For $a = 0$, put $\mathcal{C}_0 = \mathcal{L}_0 = [0, \pi]$ and $\Omega_0 = \varnothing$. We use the Fredholm essential spectrum

$$\sigma_{\text{ess}}(T) = \{\lambda : T - \lambda I \text{ is not Fredholm}\}.$$

Theorem 1.1. *The Hilbert matrix has a bounded extension to ℓ^2_α if and only if*

$$-1 < \alpha < 1. \tag{5}$$

In this range, with $a = |\alpha|$,

$$\sigma(H_\alpha) = \mathcal{L}_a, \qquad \sigma_{\text{ess}}(H_\alpha) = \mathcal{C}_a. \tag{6}$$

More precisely,

$$\sigma_p(H_\alpha) = \begin{cases} \Omega_{|\alpha|}, & -1 < \alpha < 0, \\ \varnothing, & 0 \le \alpha < 1, \end{cases} \tag{7}$$

$$\sigma_r(H_\alpha) = \begin{cases} \varnothing, & -1 < \alpha \le 0, \\ \Omega_\alpha, & 0 < \alpha < 1, \end{cases} \tag{8}$$

$$\sigma_c(H_\alpha) = \mathcal{C}_{|\alpha|}. \tag{9}$$

Every eigenvalue in (7) *is simple. For $0 < |\alpha| < 1$ and $\lambda \in \Omega_{|\alpha|}$,*

$$\operatorname{ind}(H_\alpha - \lambda I) = -\operatorname{sgn}(\alpha). \tag{10}$$

Finally,

$$r(H_\alpha) = \frac{\pi}{\cos(\pi|\alpha|/2)}. \tag{11}$$

At $\alpha = 0$, this reduces to the classical simple purely continuous spectrum $[0, \pi]$.

The proof has two independent parts. A compact comparison determines the Fredholm curve and its complementary index regions. Hill's latent eigenvectors determine which Fredholm defects are kernels and which are cokernels.

2 Diagonal conjugation and boundedness

Define the unitary map

$$U_\alpha : \ell^2_\alpha \longrightarrow \ell^2, \qquad (U_\alpha x)_n = (n+1)^{\alpha/2} x_n. \tag{12}$$

The conjugated matrix is

$$A_\alpha = U_\alpha H_\alpha U_\alpha^{-1}, \qquad (A_\alpha)_{mn} = \frac{(m+1)^{\alpha/2}(n+1)^{-\alpha/2}}{m+n+1}. \tag{13}$$

On finite sequences, and hence whenever the operators are bounded,

$$A_\alpha^* = A_{-\alpha}. \tag{14}$$

For $\alpha \ge 1$, the vector e_0 belongs to ℓ^2_α, whereas

$$\|H_\alpha e_0\|^2_\alpha = \sum_{m=0}^{\infty} (m+1)^{\alpha - 2} = \infty. \tag{15}$$

Thus the matrix is not bounded for $\alpha \ge 1$. If it were bounded for $\alpha \le -1$, then (14) would make $A_{-\alpha}$ bounded, contradicting (15). Sufficiency for $|\alpha| < 1$ follows from the comparison below.

3 A Hilbert–Schmidt comparison

Put $b = \alpha/2$, and on $L^2(1,\infty)$ define

$$(B_\alpha f)(x) = \int_1^\infty K_\alpha(x,y) f(y)\,dy, \qquad K_\alpha(x,y) = \frac{x^b y^{-b}}{x+y}. \tag{16}$$

Let $I_n = [n+1, n+2)$ and let

$$J : \ell^2 \longrightarrow L^2(1,\infty), \qquad Jx = \sum_{n\geq 0} x_n \mathbf{1}_{I_n}.$$

The operator $JA_\alpha J^*$, which is zero on the orthogonal complement of the step functions, has kernel

$$\widetilde{K}_\alpha(x,y) = \frac{(m+1)^b (n+1)^{-b}}{m+n+1}, \qquad (x,y) \in I_m \times I_n. \tag{17}$$

Lemma 3.1. *If $|\alpha| < 1$, then*

$$\widetilde{K}_\alpha - K_\alpha \in L^2\big((1,\infty)^2\big).$$

Consequently $JA_\alpha J^* - B_\alpha$ is Hilbert–Schmidt.

Proof. Write $r = m+1$ and $s = n+1$. First compare the denominator in (17) with $r+s$:

$$\left| \frac{r^b s^{-b}}{r+s-1} - \frac{r^b s^{-b}}{r+s} \right| = \frac{r^b s^{-b}}{(r+s-1)(r+s)}. \tag{18}$$

Its squared sum is bounded by

$$C \sum_{r,s\geq 1} \frac{r^\alpha s^{-\alpha}}{(r+s)^4} < \infty, \tag{19}$$

because on the shell $r+s = k$,

$$\sum_{r=1}^{k-1} r^\alpha (k-r)^{-\alpha} = O_\alpha(k)$$

exactly for $-1 < \alpha < 1$.

The mean-value theorem on $I_m \times I_n$ gives

$$|K_\alpha(x,y) - K_\alpha(r,s)| \leq C_\alpha \frac{r^b s^{-b}}{r+s} \left(\frac{1}{r} + \frac{1}{s} + \frac{1}{r+s} \right). \tag{20}$$

The square is summable. For example, the term arising from r^{-1} is

$$\sum_{r,s\geq 1} \frac{r^{\alpha-2} s^{-\alpha}}{(r+s)^2} < \infty \quad (\alpha > -1), \tag{21}$$

while the s^{-1} term requires $\alpha < 1$ and the last term is easier. Equations (18)–(21) prove the claim. $\square$

The continuous model below is bounded for $|\alpha| < 1$; hence lemma 3.1 proves the sufficiency in (5). Moreover, for every $\lambda \neq 0$,

$$A_\alpha - \lambda I \text{ is Fredholm} \iff B_\alpha - \lambda I \text{ is Fredholm}, \tag{22}$$

and the indices agree. Indeed, on the step-function subspace and its orthogonal complement, $JA_\alpha J^* - \lambda I$ is respectively unitarily equivalent to $A_\alpha - \lambda I$ and to $-\lambda I$.

4 The Wiener–Hopf symbol

The unitary map

$$V : L^2(1,\infty;dx) \longrightarrow L^2(0,\infty;du), \qquad (Vf)(u) = e^{u/2}f(e^u),$$

turns B_α into the half-line convolution operator

$$(VB_\alpha V^{-1}g)(u) = \int_0^\infty k_\alpha(u-v)g(v)\,dv, \qquad k_\alpha(s) = \frac{e^{\alpha s/2}}{2\cosh(s/2)}. \tag{23}$$

The kernel is in $L^1(\mathbb{R})$ exactly when $|\alpha| < 1$. With $\widehat{k}(t) = \int_{\mathbb{R}} e^{-its}k(s)\,ds$, the beta integral yields

$$h_\alpha(t) = \widehat{k}_\alpha(t) = \frac{\pi}{\cos(\pi\alpha/2 - i\pi t)}. \tag{24}$$

The symbol extends continuously to the one-point compactification of $\mathbb{R}$ by $h_\alpha(\infty) = 0$.

The scalar Wiener–Hopf Fredholm theorem [4, 3] therefore gives, for $\lambda \neq 0$,

$$B_\alpha - \lambda I \text{ is Fredholm} \Longleftrightarrow \lambda \notin h_\alpha(\mathbb{R} \cup \{\infty\}). \tag{25}$$

The range in (25) is precisely $\mathcal{C}_{|\alpha|}$. The parametrized curve is clockwise for $\alpha > 0$ and counterclockwise for $\alpha < 0$. We will determine the actual index directly from kernels and cokernels, avoiding any dependence on Fourier or Wiener–Hopf sign conventions.

For reference, if $\lambda = x + iy \neq 0$ lies on $\mathcal{C}_a$, then

$$(x^2+y^2)^2 = \pi^2\left(\frac{x^2}{\cos^2 c_a} - \frac{y^2}{\sin^2 c_a}\right), \qquad x > 0. \tag{26}$$

5 Hill eigenvectors and weighted membership

Hill classified all latent eigenvectors of the classical Hilbert matrix [5]; see also the modern treatments [1, 8] and the explicit formulation in [10, Theorem 4.5]. In the normalization convenient here, for $0 < \Re\mu \le 1/2$, the Taylor coefficients of

$$f_\mu(z) = (1-z)^{\mu-1}\,{}_2F_1(\mu,\mu;1;z) = \sum_{n=0}^{\infty} x_n(\mu)z^n \tag{27}$$

satisfy

$$Hx(\mu) = \frac{\pi}{\sin(\pi\mu)}x(\mu). \tag{28}$$

Every nonzero formal eigenvector of $H = H_1$ is of this form, up to a scalar and the symmetry $\mu \leftrightarrow 1-\mu$; all the corresponding root spaces are one-dimensional. In particular, all latent roots have positive real part.

For $0 < \Re\mu < 1/2$, the connection formula for ${}_2F_1$ at one and coefficient transfer give

$$x_n(\mu) = C_\mu n^{-\mu}(1+o(1)) + D_\mu n^{\mu-1}(1+o(1)), \tag{29}$$

where

$$C_\mu = \frac{\Gamma(1-2\mu)}{\Gamma(1-\mu)^3}, \qquad D_\mu = \frac{\Gamma(2\mu-1)}{\Gamma(\mu)^3}. \tag{30}$$

The first term is dominant and $C_\mu \neq 0$. Therefore

$$x(\mu) \in \ell^2_\alpha \iff 2\Re\mu > 1 + \alpha. \tag{31}$$

On the critical line $\Re\mu = 1/2$, the coefficients are $O(n^{-1/2})$ when $\mu \neq 1/2$ and $O(n^{-1/2} \log n)$ at $\mu = 1/2$. They belong to ℓ^2_α exactly when $\alpha < 0$: membership follows from these bounds, nonmembership at $\alpha = 0$ is Hill's latent-not-ℓ^2 result, and $\ell^2_\alpha \subset \ell^2$ for $\alpha > 0$.

Combining Hill's completeness with (31) gives

$$\sigma_p(H_\alpha) = \left\{ \frac{\pi}{\sin(\pi\mu)} : \frac{1+\alpha}{2} < \Re\mu \leq \frac{1}{2} \right\} = \Omega_{|\alpha|} \tag{32}$$

for $-1 < \alpha < 0$, with simple eigenvalues. For $0 \leq \alpha < 1$, there is no point spectrum.

6 Completion of the proof

Suppose first that $\lambda \notin \mathcal{L}_{|\alpha|}$ and $\lambda \neq 0$. By (22) and (25), $A_\alpha - \lambda I$ is Fredholm of index zero. Hill's classification and (31) show that it has trivial kernel. By (14), its adjoint has trivial kernel as well. Thus it is bijective, so there is no spectrum outside the lens.

If $-1 < \alpha < 0$ and $\lambda \in \Omega_{|\alpha|}$, (32) gives a one-dimensional kernel. The adjoint has positive weight and hence zero kernel, so the cokernel vanishes. The operator is surjective and has index one. If $0 < \alpha < 1$, the same argument applied to the adjoint shows that the original operator is injective with closed range of codimension one and index minus one. This proves (7), (8), and (10) in the interior.

For $\lambda \in \mathcal{C}_{|\alpha|} \setminus \{0\}$, the operator is not Fredholm. The strict threshold in (31) shows that neither it nor its adjoint has an eigenvector. It is therefore injective with dense range. The range cannot be all of the space, for a bounded bijection would be invertible and hence Fredholm. Thus every nonzero boundary point is in the continuous spectrum.

It remains to classify zero. If $H_\alpha x = 0$, then

$$f(t) = \sum_{n=0}^{\infty} x_n t^n \in L^1(0,1) \tag{33}$$

when $\alpha > -1$, since

$$\sum_{n\geq 0} \frac{|x_n|}{n+1} \leq \|x\|_\alpha \left(\sum_{n\geq 0} (n+1)^{-2-\alpha} \right)^{1/2}.$$

The equations $H_\alpha x = 0$ become

$$\int_0^1 t^m f(t)\, dt = 0 \qquad (m \geq 0). \tag{34}$$

Polynomial density implies $f = 0$ almost everywhere and hence $x = 0$. Applying the same argument to $A^*_\alpha = A_{-\alpha}$ shows that the range is dense. Nonzero boundary spectral points tend to zero. If A_α were Fredholm at zero, its dense range would be closed and hence surjective; injectivity would then make it invertible, contradicting the spectral closure. Thus zero is both continuous and Fredholm essential.

For $\alpha = 0$, the same conclusions agree with the classical results of Magnus and Rosenblum [7, 9]. This completes (6)–(9). Finally,

$$|h_\alpha(t)| = \frac{\pi}{\sqrt{\cos^2(\pi\alpha/2) + \sinh^2(\pi t)}}$$

is maximized at $t = 0$. The maximum-modulus principle on the bounded lens then yields (11) and completes the proof of theorem 1.1.

7 Remarks on scope and prior work

Silbermann determined the spectrum and Fredholm picture of the Hilbert matrix on unweighted ℓ^p and Hardy spaces [10]. Under the formal correspondence $1/p = (1+\alpha)/2$, his spectral lens is the same one that appears here, but the coefficient-weighted Hilbert space (1) is a different Banach-space realization. Jin's recent work treats boundedness of generalized Hilbert operators on weighted sequence spaces, not the spectrum computed here [6].

The compact bridge in lemma 3.1 is elementary, and an equivalent result may exist in older Wiener–Hopf or projection-method literature under different notation. Targeted searches found no theorem for the exact pair of the matrix $(m+n+1)^{-1}$ and the weight $(n+1)^{\alpha}$. Accordingly, the present note claims the stated proof, not absolute priority.

AI-assisted tools were used for source discovery, algebraic exploration, executable falsification checks, adversarial proof review, and manuscript preparation. Every mathematical claim used in the theorem is accompanied by an explicit argument or a cited theorem. The named author is responsible for the final manuscript.

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