

Low-Degree Curves and Smooth Hilbert Loci on the Fermat Fourfold

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Abstract

Let $X_d \subset \mathbf{P}_{\mathbf{C}}^5$ be the Fermat hypersurface of degree d . For $\delta \geq 2$ and $d > 10\delta - 6$, we describe the integral degree- δ Hilbert locus as a disjoint union of $15d^3$ smooth open subsets of $\mathbf{P}^{\binom{\delta+2}{2}-1}$, one for each standard coordinate-pairing plane. The point classification follows by applying Salberger's diagonal-curve inequality after a minimal-support descent. To obtain the scheme structure, we prove a three-term Wronskian lemma: on any integral plane curve of degree δ , the restriction of a standard plane's normal bundle has no sections when $d > \delta + 3$. This rules out transverse infinitesimal deformations, including at singular integral curves. The conic threshold $d \geq 15$ is sharp by Reznick's published conic on X_{14} . Combining Salberger's degree and genus inequalities also gives a stronger necessary genus bound for nonstandard curves. The result treats an explicit range of the AIM low-degree-curve programme, not its full compactified parameter spaces. No absolute priority claim is made.

1 Statement and scope

All schemes and curves in this note are over $\mathbf{C}$. Put

$$X_d = \{x_0^d + \cdots + x_5^d = 0\} \subset \mathbf{P}^5, \quad d \geq 2.$$

A curve is *integral* if it is reduced and irreducible. Its degree is measured by $\mathcal{O}(1)$, and its geometric genus is the genus of its normalization. A *standard plane* is defined by a partition of the six coordinates into three pairs $\{a_j, b_j\}$ and equations

$$x_{b_j} = \zeta_j x_{a_j}, \quad \zeta_j^d = -1, \quad 1 \leq j \leq 3.$$

There are $15d^3$ such planes: there are 15 perfect matchings, and d choices for each ζ_j . The proportionality classes of the coordinate forms on a plane recover its matching, so these are distinct.

For $\delta \geq 2$, let

$$U_\delta \subset \mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}(\delta))$$

be the open locus of irreducible forms modulo scalar. Set $p_a = (\delta - 1)(\delta - 2)/2$, and let $H_{d,\delta}^{\text{int}}$ be the geometrically integral open locus in $\text{Hilb}^{\delta t + 1 - p_a}(X_d)$.

Theorem 1.1. *If $\delta \geq 2$ and $d > 10\delta - 6$, every integral degree- δ curve on X_d lies in a unique standard plane. In particular, it has arithmetic genus p_a . Moreover there is an isomorphism of schemes*

$$H_{d,\delta}^{\text{int}} \simeq \coprod_{15d^3 \text{ standard planes}} U_\delta.$$

Thus this Hilbert locus is smooth and reduced, and each of its components has dimension $\binom{\delta+2}{2} - 1$.

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For $\delta = 2$, the condition is $d \geq 15$, and each component is $\mathbf{P}^5$ minus the cubic discriminant of singular conics. This cutoff cannot be decreased, by the known example recalled in Section 5. An integral curve with a different Hilbert polynomial but the same degree cannot occur in the range of the theorem, by its point classification.

The motivating question, recorded as Question 26 of the AIM workshop report [1], asks for the variety of low-degree curves on the six-coordinate Fermat hypersurface. It leaves both “low degree” and the parameter-space convention unspecified. We give an explicit degree range and a Hilbert-scheme statement for integral curves. We do not describe its compactification at reducible or nonreduced curves, the remaining degree ranges, or stable-map spaces.

The point classification combines Salberger’s published Theorem 9.1 [2] with grouping proportional coordinates and choosing a minimal support. This reduction also appears in the research aid for entry `AIM-ALGEBRAIC_NUMBER_THEORY-0111` of the frozen `ulamai/UnsolvedMath` v1.6.0 dataset [5]. Our additional argument concerns the scheme structure: a direct normal-section vanishing lemma promotes the point classification to Theorem 1.1. These ingredients are classical in method; the presentation makes no claim of absolute novelty or priority. For previous line classifications, see Deland [4, Section 2.4.2]. Lines are excluded here, since their coordinate blocks need not be pairs.

2 Minimal diagonal support

We recall the exact form of the published inequality we use. If an integral curve $D \subset \mathbf{P}^{m-1}$, of degree Δ and geometric genus h , lies on exactly one degree- d diagonal hypersurface up to scalar, with every coefficient nonzero, then

$$d \leq \frac{m-2}{2}((m-1)\Delta - m + 3), \quad (1)$$

$$d \leq m(m-2) + \frac{(m-1)(m-2)(h-1)}{\Delta}. \quad (2)$$

These are rearrangements of Salberger’s Theorem 9.1(b) and (a), respectively. The uniqueness hypothesis is important.

Proposition 2.1. *Every integral curve $C \subset X_d$ of degree $\delta \geq 2$ with $d > 10\delta - 6$ lies in a unique standard plane.*

Proof. Let $\nu : \tilde{C} \rightarrow C$ be the normalization, and let $L = \nu^*\mathcal{O}_C(1)$, of degree δ . Write $s_i = \nu^*x_i$. Discard identically zero sections, and group the others by proportionality. Choose representatives t_j and write $s_i = c_i t_j$ in block J_j . Then, for $A_j = \sum_{i \in J_j} c_i^d$, the Fermat equation becomes

$$\sum_j A_j t_j^d = 0. \quad (3)$$

Suppose some A_j is nonzero. Choose an inclusion-minimal linearly dependent subfamily of the powers t_j^d , of size m , and write its relation as $\sum_{j=1}^m b_j t_j^d = 0$. Every b_j is nonzero, and the relation is unique up to scalar: two independent relations would give one on a proper subfamily after cancelling a coefficient. Also $3 \leq m \leq 6$. A two-term relation would make $(t_i/t_j)^d$ constant, and hence t_i/t_j constant, contrary to the choice of blocks.

Remove the common zero divisor B of these m sections. They then give a base-point-free, nonconstant morphism to an integral image $D \subset \mathbf{P}^{m-1}$. If e is its generic degree, then

$$e\Delta = \deg L(-B) = \delta - \deg B \leq \delta.$$

Any second independent diagonal relation on D would pull back to a second relation among the chosen powers. Thus (1) applies. Its bounds for $m = 3, 4, 5, 6$ are

$$\Delta, \quad 3\Delta - 1, \quad 6\Delta - 3, \quad 10\Delta - 6.$$

Their maximum for $\Delta \geq 1$ is the last, contradicting $d > 10\delta - 6$.

It follows that every A_j in (3) is zero. Each block has at least two coordinates, so there are at most three blocks. Since an integral curve of degree at least two does not span a line, its coordinate sections span at least three dimensions, so there are at least three blocks. Hence there are exactly three two-coordinate blocks, and no zero coordinates. Their internal cancellations define a standard plane. The nonlinear curve spans this plane, which proves uniqueness. $\square$

It is essential here to choose a minimal *linear dependence*, not just a minimal zero subsum of the given equation. The base divisor and generic degree are retained so that no birationality assumption is hidden.

3 A Wronskian vanishing lemma

Lemma 3.1. *Let $C \subset \mathbf{P}^2$ be an integral curve of degree $\delta \geq 2$, with normalization $\tilde{C}$ of genus g . Put $L = \nu^* \mathcal{O}_C(1)$ and write x, y, z for its coordinate sections. Suppose that an integer e satisfies*

$$e > \delta, \quad (e - 5)\delta > 2g - 2.$$

Then the map

$$H^0(\tilde{C}, L)^3 \longrightarrow H^0(\tilde{C}, L^{e+1}), \quad (h_x, h_y, h_z) \longmapsto h_x x^e + h_y y^e + h_z z^e$$

is injective. In particular it is injective if $e > \delta + 2$.

Proof. The sections x, y, z are nonzero and pairwise nonproportional, since C is not a line. A relation with just two nonzero terms would give $(x/y)^e = -h_y/h_x$, after renaming coordinates. The nonconstant function x/y has a zero or pole of nonzero integer order. The left side then has absolute order at least e , whereas the right side has absolute order at most δ : every nonzero section of L has a zero divisor of degree δ . This is impossible. A one-term relation is impossible as well.

Assume all three h_i are nonzero. If two coordinate sections vanished at a point, the third would be nonzero because the coordinates have no common zero. The relation would then force the third h_i to vanish to order at least $e > \delta$, again impossible. The divisors of zeroes of x, y, z are therefore pairwise disjoint under this assumption.

Put $f_i = h_i x_i^e \in H^0(\tilde{C}, L^{e+1})$. No two f_i are proportional, by the preceding quotient argument. They span a two-dimensional space because their sum is zero. In characteristic zero,

$$W = f_x df_y - f_y df_x \in H^0(\tilde{C}, L^{2(e+1)} \otimes K_{\tilde{C}})$$

is nonzero. The derivative-of-frame terms cancel under a change of local trivialization, so this is a global section. Any pair among the f_i has the same Wronskian up to sign.

At a zero of x_i of order $a \geq 1$, use a pair containing f_i . It gives $\text{ord } W \geq ea - 1$. If r is the total number of distinct zeroes of x, y, z , their disjointness gives

$$2(e + 1)\delta + 2g - 2 = \deg \text{div}(W) \geq 3e\delta - r \geq 3(e - 1)\delta.$$

This implies $(e - 5)\delta \leq 2g - 2$, a contradiction. Finally $2g - 2 \leq \delta(\delta - 3)$ by the plane arithmetic genus, so $e > \delta + 2$ suffices. $\square$

The normalization handles singular curves. Multiple coordinate zeroes or zeroes of the h_i do not invalidate the estimates; they can only increase their orders. The integrality and characteristic-zero assumptions are used essentially in the quotient and Wronskian arguments.

4 Transverse rigidity and the Hilbert scheme

Proposition 4.1. *Let $P \subset X_d$ be a standard plane and $C \subset P$ any integral degree- δ curve with $\delta \geq 2$. If $d > \delta + 3$, then*

$$H^0(C, N_{P/X_d}|_C) = 0, \quad h^0(C, N_{C/X_d}) = \binom{\delta+2}{2} - 1.$$

The integral fixed-plane family is locally an open component of the integral Hilbert locus at $[C]$.

Proof. Use coordinates x, y, z on P , one from each coordinate pair. The normal sequence for $P \subset X_d \subset \mathbf{P}^5$ is

$$0 \longrightarrow E = N_{P/X_d} \longrightarrow \mathcal{O}_P(1)^3 \xrightarrow{\phi} \mathcal{O}_P(d) \longrightarrow 0. \quad (4)$$

After rescaling the three transverse directions, $\phi(h_x, h_y, h_z) = x^{d-1}h_x + y^{d-1}h_y + z^{d-1}h_z$. Indeed, differentiating the Fermat equation in the normal coordinate $x_b - \zeta x_a$ gives a nonzero constant times x_a^{d-1} . The entries have no common zero, so the map is surjective and E is locally free of rank two.

A section of $E|_C$ pulls back injectively to the normalization, because C is integral and the bundle is locally free. It becomes a relation in Lemma 3.1 with $e = d - 1 > \delta + 2$. This proves the vanishing. The embeddings in the chain $C \subset P \subset X_d$ are regular, even if C is singular. Their normal sequence is therefore

$$0 \longrightarrow \mathcal{O}_C(\delta) \longrightarrow N_{C/X_d} \longrightarrow E|_C \longrightarrow 0. \quad (5)$$

Taking sections and using $0 \rightarrow \mathcal{O}_P \rightarrow \mathcal{O}_P(\delta) \rightarrow \mathcal{O}_C(\delta) \rightarrow 0$ gives the claimed dimension $N = \binom{\delta+2}{2} - 1$.

The Hilbert tangent space at $[C]$ is $H^0(N_{C/X_d})$. The fixed-plane family U_δ is smooth of dimension N , and its immersion into the integral Hilbert locus is closed: containment in the fixed closed subscheme P is a closed Hilbert condition. Thus the ambient local ring at $[C]$ has dimension at least N and embedding dimension N , and is a regular local ring of dimension N . Its surjection to the local ring of U_δ has zero kernel, since a nonzero ideal in a regular local domain lowers dimension. The closed immersion is consequently an isomorphism on a neighborhood of $[C]$, by coherence of its ideal. $\square$

Proof of Theorem 1.1. Proposition 2.1 puts every integral degree- δ curve in a unique standard plane. Its arithmetic genus is the plane genus. For $\delta \geq 2$, the inequality $d > 10\delta - 6$ implies $d > \delta + 3$, so Proposition 4.1 applies at every closed point of the integral Hilbert locus. The finitely many fixed-plane families are disjoint, and each is locally an isomorphism onto that locus. A nonempty closed subset of a finite-type complex scheme has a closed point, so there are no extra points or residual nilpotent structure. The displayed disjoint-union map is an isomorphism. $\square$

This argument does not identify the closures of these families at nonintegral curves. It also does not use the invalid implication that a classification of closed points alone rules out nilpotents; the normal-section calculation is what supplies that missing information.

5 The conic boundary

Let ϵ be a primitive fifth root of unity and put

$$f_j(u, v) = \epsilon^j u^2 + iuv + \epsilon^{-j} v^2 \quad (0 \leq j \leq 4), \quad f_5(u, v) = \sqrt{-5} uv.$$

Reznick's equations (4.16)–(4.17) [3] give

$$\sum_{j=0}^5 f_j(u, v)^{14} = 0. \quad (6)$$

This identity and its corresponding nonstandard conic are inherited prior art, not a new counterexample of this note.

The forms have no common zero, since $f_5 = 0$ forces $uv = 0$, where the others are nonzero. Their span is all of $H^0(\mathbf{P}^1, \mathcal{O}(2))$: f_5 supplies uv , and

$$\sum_{j=0}^4 \epsilon^{-j} f_j = 5u^2, \quad \sum_{j=0}^4 \epsilon^j f_j = 5v^2.$$

Thus $[u : v] \mapsto [f_0 : \cdots : f_5]$ is the quadratic Veronese embedding followed by a linear plane embedding. Its image is a smooth conic on X_{14} . The six coordinate forms are pairwise nonproportional, whereas on a standard plane they occur in three proportional pairs. This conic is nonstandard, proving sharpness of the cutoff $d \geq 15$.

6 A combined genus bound

The degree and genus inequalities yield more together than when the image degree in (2) is discarded.

Corollary 6.1. *Let $d \geq 24$ and let $C \subset X_d$ be an integral curve of degree at least two, not contained in a standard plane. If g is its geometric genus, then*

$$g \geq 1 + \left\lceil \frac{(d-24)\lceil (d+6)/10 \rceil}{20} \right\rceil.$$

In particular, every nonlinear rational curve is standard-plane for $d \geq 24$, every genus-one curve is standard-plane for $d \geq 25$, and every genus-two curve is standard-plane for $d \geq 30$.

Proof. If every coordinate block cancels internally, the last paragraph of Proposition 2.1 already makes C standard-plane. Otherwise construct the minimal-support image D , with degree Δ and genus h , as above. Its normalization is dominated by that of C , so $h \leq g$. The maximum of the four bounds in (2), for $3 \leq m \leq 6$, is $24 + 20(h-1)/\Delta$. For $h \geq 1$ this is immediate. For $h = 0$, the differences from the other three bounds are $21 - 18/\Delta$, $16 - 14/\Delta$, and $9 - 8/\Delta$, all positive. Together with (1) this gives

$$\Delta \geq \lceil (d+6)/10 \rceil, \quad 20(h-1) \geq (d-24)\Delta.$$

Since $d-24 \geq 0$, substitution and integrality give the assertion. The examples follow directly, and the bound is nondecreasing with d . $\square$

No optimality or existence at this necessary genus bound is asserted. The genus-only conclusion does not give the scheme-level rigidity without the additional degree condition $d > \delta + 3$.

7 Verification and remaining questions

The universal assertions rest on the preceding written arguments. The accompanying standard-library Python checker expands (6) exactly with the fifth-root filter: the first five terms sum to $5^7(uv)^{14}$, cancelled by f_5^{14} . It also tests the normal-section map on all 246 independent triples of sign-normalized quadratics with coefficients in $\{-1, 0, 1\}$, at four exponents. This gives 1,968 exact full-rank computations over two finite fields, with 60 additional rational computations. Another 88 rational rank computations treat smooth Fermat plane curves and irreducible cuspidal plane curves of degrees 2 through 12. Inequality checks and hypothesis-failure examples are recorded separately. The same results reproduce in isolated Python 3.9 and 3.13 environments. These are finite diagnostics, not

independent mathematical review or formal verification, and do not establish the universal claims by themselves.

The boundary $d = 10\delta - 6$ beyond the known conic, curve degrees outside our range, and intersections of closures through nonintegral curves remain untreated. A bounded literature search cannot exclude folklore or other prior formulations of these consequences. This preprint is unrefereed. AI assistance was used in developing and checking the arguments, searching primary sources, implementing the diagnostics, and preparing the text; responsibility for the submitted claims and their stated scope remains with the author.

References

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