

Semisimple Torsion Fields of Modular Jacobians at Powers of 11 and 13

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Abstract

For $\ell \in \{11, 13\}$ and every $m \geq 1$, we identify the semisimple ℓ -torsion field of the Jacobian of the connected principal modular curve $X(\ell^m)$ over $K_m = \mathbb{Q}(\zeta_{\ell^m})$. It is the compositum of K_m with the field cut out by the residual Ramanujan Delta representation, and its Galois group over K_m is $\mathrm{SL}_2(\mathbb{F}_\ell)$. The full torsion field has an ℓ -group kernel over this quotient; the prime-to- ℓ part of its degree is exactly 120 or 168, respectively. The result is an explicit structural corollary of level-one modularity, the modular decomposition of Jacobians, and Hoshi–Iijima’s non-pro- ℓ theorem. We give a cyclotomic-field moduli construction of the covering needed in the argument and reproducible exact certificates for the two residual Delta images. We do not determine the full torsion field or claim to resolve the unrestricted AIM splitting-field question. This note is an unrefereed synthesis; absolute priority is not asserted.

1 Statement and scope

Let $\ell \in \{11, 13\}$, $m \geq 1$, $N = \ell^m$, and $K_m = \mathbb{Q}(\zeta_N)$, with a fixed primitive N th root of unity. Write $X(N)/K_m$ for the connected principal modular curve whose noncuspidal points classify elliptic curves with ordered bases (Q, R) of their N -torsion satisfying $e_N(Q, R) = \zeta_N$. Set

$$J_m = \mathrm{Jac} X(N), \quad g_m = \dim J_m, \quad L_m = K_m(J_m[\ell]).$$

For a number field K , put $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. Let L_m^{ss} be the fixed field of the kernel of the semisimplification of the G_{K_m} -module $J_m[\ell]$. Also put

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n, \quad \bar{\rho}_{\Delta, \ell} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F}_\ell),$$

and let M_ℓ be the field fixed by $\ker \bar{\rho}_{\Delta, \ell}$. All composita are taken inside the same algebraic closure.

Theorem 1. *For every $\ell \in \{11, 13\}$ and $m \geq 1$,*

$$L_m^{\mathrm{ss}} = K_m M_\ell, \quad \mathrm{Gal}(L_m^{\mathrm{ss}}/K_m) \simeq \mathrm{SL}_2(\mathbb{F}_\ell).$$

There is an exact sequence

$$1 \longrightarrow U_m \longrightarrow \mathrm{Gal}(L_m/K_m) \longrightarrow \mathrm{SL}_2(\mathbb{F}_\ell) \longrightarrow 1$$

with U_m an ℓ -group. In particular,

$$[L_m : K_m] = (\ell^3 - \ell) \ell^{u_m}, \quad 0 \leq u_m \leq g_m^2 - 1.$$

The prime-to- ℓ part of this degree is $\ell^2 - 1$: it is 120 for $\ell = 11$ and 168 for $\ell = 13$.

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Hall's open-ended question in the AIM workshop problem list [1, item 14, p. 4] asks for information about a modular Jacobian's torsion splitting field. Its brief formulation leaves the base field and modular-curve convention unspecified. Our principal-curve convention and cyclotomic base are explicit above; the theorem does not answer that question in full.

The main external inputs are the newform decomposition of modular Jacobians, Khare's level-one modularity theorem [3], and Hoshi–Iijima's classification of pro- ℓ images [2]. These theorems, and the classical large image of Delta, are not new results of this note. The contribution presented here is their explicit combination into a uniform semisimple-field description.

2 A covering defined over the cyclotomic field

Lemma 2. *For $N \geq 3$, there is a finite nonconstant morphism $X_1(N^2)_{\mathbb{Q}(\zeta_N)} \longrightarrow X(N)$ with the convention above.*

Proof. On the open modular curves, start with (E, P) , where P has exact order N^2 . Let $A = \langle NP \rangle$, let $\phi : E \longrightarrow E' = E/A$ be the cyclic N -isogeny, and let $\hat{\phi} : E' \longrightarrow E$ be its dual. The point $Q = \phi(P)$ has exact order N , and $\hat{\phi}(Q) = NP$. The cyclic group $B = \ker \hat{\phi}$ has order N and lies in $E'[N]$. Moreover, the induced map $E'[N]/B \longrightarrow A$ is an isomorphism, and Q generates this quotient. If $aQ \in B$, then $aNP = 0$, so $N \mid a$ and $aQ = 0$. Thus $\langle Q \rangle \oplus B = E'[N]$.

The restriction $e_N(Q, -) : B \longrightarrow \mu_N$ is an isomorphism. Indeed, a generator of B together with Q is a basis of the free rank-two $(\mathbb{Z}/N\mathbb{Z})$ -module $E'[N]$, so their Weil pairing is primitive. There is therefore a unique $R \in B$ with $e_N(Q, R) = \zeta_N$. This construction commutes with base change over $\mathbb{Q}(\zeta_N)$ and defines a morphism on the moduli schemes, sending (E, P) to (E', Q, R) .

The morphism is nonconstant: for a fixed target elliptic curve E' , only finitely many source isomorphism classes E can occur, since the dual isogeny is specified by a cyclic subgroup of $E'[N]$. A constant target would contradict the nonconstant j -map of $X_1(N^2)$. The morphism extends to the smooth proper compactifications, where a nonconstant morphism of curves is finite and surjective. $\square$

Analytically this agrees with the congruence-subgroup inclusion

$$\Gamma_1(N^2) \subset \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma(N) \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix};$$

compare [6, §11.1]. The algebraic construction, not the analytic inclusion alone, proves the required descent.

3 Residual composition factors

For a finite map $f : C \rightarrow D$ of smooth proper curves, pullback and pushforward on Jacobians satisfy $f_* f^* = [\deg f]$. After tensoring Tate modules with $\mathbb{Q}_\ell$, this makes $V_\ell(\text{Jac } D)$ a direct summand of $V_\ell(\text{Jac } C)$. The conclusion holds even if ℓ divides $\deg f$; it is not an assertion of an injection on integral ℓ -torsion.

We recall the lattice justification for passing to residual composition factors. If $\ell\Lambda \subset \Lambda' \subset \Lambda$ are stable lattices, multiplication by ℓ gives $\Lambda/\Lambda' \simeq \ell\Lambda/\ell\Lambda'$. The exact sequences for the two reductions then show that $\Lambda/\ell\Lambda$ and $\Lambda'/\ell\Lambda'$ have the same composition factors. Scaling, intersections, and finite chains connect arbitrary commensurable stable lattices. The same proof uses a uniformizer after extending coefficients. The intersection of a rational subrepresentation with a stable lattice

is saturated, so its reduction injects into the reduction of that lattice. Consequently, by Lemma 2, every composition factor of $J_m[\ell] \otimes \overline{\mathbb{F}}_\ell$ occurs in the restriction of $J_1(N^2)[\ell]$ to G_{K_m} .

The newform decomposition of $J_1(N^2)$ is defined over $\mathbb{Q}$. Its factors, up to isogeny and multiplicity, are the abelian varieties attached to weight-two newforms of levels dividing N^2 . After a finite extension of $\mathbb{Q}_\ell$, their Tate modules split into two-dimensional newform representations, odd and unramified away from ℓ ; see [4, §§14.2–14.3] and [5, Theorem 3.1]. Over $\overline{\mathbb{F}}_\ell$, their residual factors are therefore either one-dimensional characters or two-dimensional absolutely irreducible odd representations, all unramified away from ℓ . In the irreducible case, oddness follows from determinant -1 at complex conjugation, since ℓ is odd.

Every one-dimensional character here is a power of the mod- ℓ cyclotomic character $\bar{\chi}_\ell$. Its finite image has order prime to ℓ , and Kronecker–Weber places its field in the ℓ -power cyclotomic tower. The prime-to- ℓ quotient of this tower factors through $(\mathbb{Z}/\ell\mathbb{Z})^\times$.

For a two-dimensional irreducible factor σ , a cyclotomic twist has Serre weight in $[2, \ell + 1]$. Khare’s level-one theorem realizes this twist in the characteristic-zero cusp space of that weight; the weight reduction is stated at the beginning of [3, §2], and the modularity statement is [3, Theorem 1.1]. This invokes the stated modularity theorem, not an assertion that arbitrary characteristic- ℓ modular forms lift.

For $\ell = 11$ or 13 , the only nonzero level-one cusp space in the indicated range is $S_{12} = \mathbb{C}\Delta$. In particular, $S_{14} = \Delta M_2 = 0$; see the classical level-one structure in [4, Chapters 3–4]. Thus every two-dimensional factor is

$$\bar{\rho}_{\Delta, \ell} \otimes \bar{\chi}_\ell^a \quad \text{for some integer } a. \quad (1)$$

4 Exact certificates for the Delta images

The integral ℓ -adic representation of Δ has determinant χ_ℓ^{11} and Frobenius polynomial $X^2 - \tau(p)X + p^{11}$ at a prime $p \neq \ell$. At $p = 2$, the discriminant is $(-24)^2 - 4 \cdot 2^{11} = -7616$. Its residues, 7 modulo 11 and 2 modulo 13, are nonsquares. Hence the residual image has no invariant $\mathbb{F}_\ell$ -line.

For a second witness let $d_p = \tau(p)^2 - 4p^{11}$. Exact values are

ℓ	p	$\tau(p)$	$d_p \bmod \ell^2$
11	3	252	88
13	103	-225755128648	52

Here $d_3 = -645084$ and $d_{103} = -4403976718102792816284$, so both discriminants have ℓ -adic valuation exactly one. A matrix scalar modulo ℓ has the form $\lambda I + \ell B$; its discriminant $(a-d)^2 + 4bc$ is divisible by ℓ^2 . Our Frobenius matrices are consequently nonscalar modulo ℓ with repeated nonzero eigenvalues. A power of each is a nonidentity unipotent.

An irreducible subgroup $H \leq \mathrm{GL}_2(\mathbb{F}_\ell)$ containing such a unipotent contains $\mathrm{SL}_2(\mathbb{F}_\ell)$. Indeed, some element of H moves its fixed line; conjugation yields another unipotent with a different fixed line. In a basis from those two lines, they are upper and lower elementary matrices with nonzero parameters. Their powers give every parameter in $\mathbb{F}_\ell$, because this is a prime field. Upper and lower elementary matrices generate $\mathrm{SL}_2(\mathbb{F}_\ell)$. Finally, $\gcd(11, \ell - 1) = 1$, so the determinant $\bar{\chi}_\ell^{11}$ is surjective. We have proved

$$\mathrm{Gal}(M_\ell/\mathbb{Q}) = \mathrm{GL}_2(\mathbb{F}_\ell). \quad (2)$$

The accompanying scripts compute the first 200 coefficients by the Euler product, its logarithmic derivative, and $(E_4^3 - E_6^2)/1728$. These three exact methods agree and certify the displayed witnesses. The group argument above, not a finite experiment alone, establishes (2).

5 The cyclotomic restriction and the field identity

For $\ell \geq 5$, the group $\mathrm{SL}_2(\mathbb{F}_\ell)$ is perfect. Choose $a \in \mathbb{F}_\ell^\times$ with $a^2 \neq 1$. Commuting $\mathrm{diag}(a, a^{-1})$ with an upper elementary matrix multiplies its parameter by $a^2 - 1$, so every upper elementary matrix is a commutator. The same applies to lower ones. Thus the commutator subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$ is exactly $\mathrm{SL}_2(\mathbb{F}_\ell)$.

By the determinant calculation, the maximal abelian subextension of $M_\ell/\mathbb{Q}$ is $K_1 = \mathbb{Q}(\zeta_\ell)$. Since $K_m/\mathbb{Q}$ is abelian and contains K_1 ,

$$M_\ell \cap K_m = K_1, \quad \bar{\rho}_{\Delta, \ell}(G_{K_m}) = \mathrm{SL}_2(\mathbb{F}_\ell).$$

The standard module of this group is absolutely irreducible: over any extension field, an invariant line for all upper and lower elementary matrices would have to be both coordinate lines.

All cyclotomic characters in Section 3 become trivial on G_{K_m} . It follows from (1) that every composition factor of $J_m[\ell] \otimes \bar{\mathbb{F}}_\ell$ is either trivial or the restricted standard Delta module. Here the restricted factors have each been proved simple; we have not assumed that restriction preserves semisimplicity in modular characteristic.

Hoshi-Iijima [2, Theorem A, Theorem 4.1] show that the ℓ -adic image of J_m over K_m is not pro- ℓ for $\ell = 11, 13$. If all its residual factors were trivial, a basis adapted to a composition series over a finite extension of $\mathbb{F}_\ell$ would make the residual image upper unitriangular, hence an ℓ -group. The kernel of reduction of the integral ℓ -adic image is pro- ℓ , so the whole image would then be pro- ℓ , a contradiction. At least one standard Delta factor therefore occurs.

The kernel of a direct sum of a positive number of copies of this factor, with any trivial factors, is precisely $\ker(\bar{\rho}_{\Delta, \ell}|_{G_{K_m}})$. Taking fixed fields proves the field identity and the semisimple Galois group in Theorem 1. The semisimple action factors through the full torsion image. Its kernel acts trivially on every factor of a composition series, so the same unitriangular argument makes this kernel U_m an ℓ -group.

The principal polarization and $\mu_\ell \subset K_m$ put the full residual image inside $\mathrm{Sp}_{2g_m}(\mathbb{F}_\ell)$. The ℓ -part of its order is at most $\ell^{g_m^2}$, whereas $|\mathrm{SL}_2(\mathbb{F}_\ell)| = \ell(\ell^2 - 1)$. Writing $|U_m| = \ell^{u_m}$ gives $0 \leq u_m \leq g_m^2 - 1$ and all remaining conclusions of the theorem.

6 Limitations and reproducibility

The extension L_m/L_m^{ss} , its exact degree, and its extension classes are not computed. The semisimple quotient itself has order divisible by ℓ : the phrase “prime-to- ℓ part” refers to an integer degree, not to a prime-to- ℓ Galois extension. At other primes, additional small-weight cusp systems can occur, so this argument does not give the same single-field description.

The source archive includes the proof dossier and portable standard-library Python checkers, with recorded normal and optimized-mode outputs. Besides the three coefficient constructions, these check elementary group orders and scalar-lift discriminants. They also retain the control that a nontrivial unipotent module and its trivial semisimplification can have the same characteristic polynomials. These checks are not a formal proof or an independent review of the argument. Third-party source PDFs are not redistributed in the archive.

AI assistance was used in literature retrieval, development and self-audit of the argument, exact computations, and preparation of this manuscript. The author is responsible for the claims and citations. This unrefereed note makes no absolute-priority claim and no claim to close the general source problem.

References

- [1] American Institute of Mathematics, *The Tate conjecture: problem list*, workshop of 23–27 July 2007, transcribed by Christopher Lyons, item 14 (C. Hall), p. 4. <https://aimath.org/WWN/tateconjecture/tateconjecture.pdf>.
- [2] Y. Hoshi and Y. Iijima, The pro- l outer Galois actions associated to modular curves of prime power level, *Journal de Théorie des Nombres de Bordeaux* **30** (2018), 781–799. <https://doi.org/10.5802/jtnb.1049>.
- [3] C. Khare, *On Serre’s modularity conjecture for 2-dimensional mod p representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside p* , preprint (2005), Theorem 1.1 and Sections 2, 7.2. <https://arxiv.org/abs/math/0504080>.
- [4] K. A. Ribet and W. A. Stein, *Lectures on Modular Forms and Hecke Operators*, 5 October 2011, Chapters 3–4 and Sections 14.2–14.3. https://wstein.org/books/ribet-stein/marked_up/2011-10-06-ch6.pdf.
- [5] P. Bruin, *Modular curves and Galois representations*, Beirut lecture notes, Theorem 3.1. <https://pub.math.leidenuniv.nl/~bruinpj/Beirut.pdf>.
- [6] A. V. Sutherland, *18.786 Number Theory II, Lecture 11*, MIT, 18 March 2024, Section 11.1. <https://math.mit.edu/classes/18.786/2024/LectureNotes11.pdf>.